TRIGONOMETRICALIZABLE QUINTIC EQUATION

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This note discusses on a specific class of quintic equations which are solvable thanks to the multiple-angle trigonometric transformation. Besides, the author finds an interesting fact, namely "Matryoshka paradox" where quintic equations are more convenient to solve than an order of generalized quartic, resolvent cubic, and factoring quadratic equations.

Initiation

This note focuses on the specific reduced form of quintic equations:

$$x^5 + ax^3 + bx + c = 0$$

Except for the constant coefficient, the above equation contains odd-ordered parameters. Hence, the methodology is based on the expanding formula:

$$\cos 5\theta = 16\cos\theta^5 - 20\cos\theta^3 + 5\cos\theta$$

Replacing $x = k\cos\theta$ (of which $k \neq 0$), the equation becomes:

$$(k\cos\theta)^5 + a(k\cos\theta)^3 + b(k\cos\theta) + c = 0$$

Hence, we shall find k such that:

$$(k^5 : ak^3 : bk) = (k^4 : ak^2 : b) = (16 : -20 : 5)$$

Accordingly, we get:

$$\begin{cases} \dfrac{k^2}{a} = -\dfrac{4}{5} \\ \dfrac{ak^2}{b} = -4 \end{cases} \Leftrightarrow \begin{cases} a = -\dfrac{5k^2}{4} = -5\lambda \\ b = \dfrac{5k^4}{16} = 5\lambda^2 \end{cases} \quad \left(\lambda = \frac{k^2}{4}\right)$$

In other words, the above mentioned quintic equation is solvable thanks to this method if there exists λ such that $a = -5\lambda$ and $b = 5\lambda^2$. Amazingly, λ may be negative and even complex numbers. Illustrated examples shall be discussed in the next section.

Illustration

Example 1. Solve the equation: $x^5 - 15x^3 + 45x - 27 = 0$

Following the finding in the previous section, we shall find λ such that $-15 = -5\lambda$ and $45 = 5\lambda^2$. This results in $\lambda = 3$ and therefore, $k = 2\sqrt{\lambda} = 2\sqrt{3}$.

Replacing $x = 2\sqrt{3}\cos\theta$, the equation becomes:

$$18\sqrt{3}(16\cos\theta^5 - 20\cos\theta^3 + 5\cos\theta) = 27 \Leftrightarrow \cos 5\theta = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$$

Solving this elementary trigonometric equation, we get:

$$5\theta = \pm\frac{\pi}{6} + \mu 2\pi \Leftrightarrow \theta = \pm\frac{\pi}{30} + \frac{\mu 2\pi}{5} \quad (\mu \in \mathbb{Z})$$

Consequently, solutions for the given quintic equation are:

$$x = 2\sqrt{3}\cos\left(\frac{\pi}{30} + \frac{\nu 2\pi}{5}\right) \quad (\nu \in \{-2; -1; 0; 1; 2\})$$

Particularly, the above solutions could be expressed as follows:

$$x_1 = 2\sqrt{3}\cos\left(-\frac{23\pi}{30}\right) = 2\sqrt{3}\cos(-138°) = \frac{1}{4}\left(3 - 3\sqrt{5} - \sqrt{30 + 6\sqrt{5}}\right)$$

$$x_2 = 2\sqrt{3}\cos\left(-\frac{11\pi}{30}\right) = 2\sqrt{3}\cos(-66°) = \frac{1}{4}\left(3 + 3\sqrt{5} - \sqrt{30 - 6\sqrt{5}}\right)$$

$$x_3 = 2\sqrt{3}\cos\left(\frac{\pi}{30}\right) = 2\sqrt{3}\cos(6°) = \frac{1}{4}\left(3 + 3\sqrt{5} + \sqrt{30 - 6\sqrt{5}}\right)$$

$$x_4 = 2\sqrt{3}\cos\left(\frac{13\pi}{30}\right) = 2\sqrt{3}\cos(78°) = \frac{1}{4}\left(3 - 3\sqrt{5} + \sqrt{30 + 6\sqrt{5}}\right)$$

$$x_5 = 2\sqrt{3}\cos\left(\frac{5\pi}{6}\right) = 2\sqrt{3}\cos(150°) = -3$$

Example 2. Solve the equation: $x^5 + 5x^3 + 5x + 2\xi = 0$
Regarding this problem, $\lambda = -1 < 0$ gives us $k = 2i$, of which $i^2 = -1$.

Replacing $x = 2i\cos\theta$, the equation becomes:

$$2i\cos 5\theta = -2\xi \Leftrightarrow \cos 5\theta = \xi i \Leftrightarrow 5\theta = \pm\cos^{-1}\xi i + \mu 2\pi \Leftrightarrow \theta = \pm\frac{1}{5}\cos^{-1}\xi i + \frac{\mu 2\pi}{5} \quad (\mu \in \mathbb{Z})$$

Finally, solutions for the given quintic equation are:

$$x_\nu = 2i\cos\left(\frac{1}{5}\cos^{-1}\xi i + \frac{\nu 2\pi}{5}\right) \quad (\nu \in \{-2; -1; 0; 1; 2\})$$

The above expression contains an inversed trigonometric function of complex numbers. Interestingly, according to Bézout's theorem for odd-ordered polynomials, there exists $\nu \in \{-2; -1; 0; 1; 2\}$ such that $x_\nu \in \mathbb{R}$.

Illusion
In the equation as discussed in Example 1, it is observable that $x = -3$ is a comfortable solution. This section re-solves Example 1 in a purely algebraic method and then compares the two methodologies. Interestingly, the trigonometricalizable quintic polynomial seems more convenient.

Factoring the quintic polynomial as mentioned in Example 1, we get:

$$x^5 - 15x^3 + 45x - 27 = (x + 3)(x^4 - 3x^3 - 6x^2 + 18x - 9)$$

Considering the generalized quartic equation:

$$x^4 + 4\alpha x^3 + \beta x^2 + \gamma x + \delta = 0$$

By replacing $x = y - \alpha$, we get the reduced form quartic equation:

$$y^4 + py^2 + qy + r = 0$$

Where p, q, and r is respectively determined as follows:

$$\begin{cases} p = \beta - 6\alpha^2 \\ q = \gamma - 2\beta\alpha + 8\alpha^3 \\ r = \delta - \gamma\alpha + \beta\alpha^2 - 3\alpha^4 \end{cases}$$

Next, we shall find two quadratic polynomials that their product is exactly the above reduced form. Accordingly, it is supposed to find the triplet (u, v, w) such that:

$$y^4 + py^2 + qy + r \equiv (y^2 + uy + v)(y^2 - uy + w)$$

As coefficients from both sides are homogeneous, we get:

$$\begin{cases} -u^2 + v + w = p \\ u(w - v) = q \\ vw = r \end{cases} \Leftrightarrow \begin{cases} v = \frac{1}{2}\left(u^2 + p - \frac{q}{u}\right) \\ w = \frac{1}{2}\left(u^2 + p + \frac{q}{u}\right) \\ vw = r \end{cases}$$

The resolvent cubic equation is obtained through the following process:

$$\frac{1}{4}\left(u^2 + p - \frac{q}{u}\right)\left(u^2 + p + \frac{q}{u}\right) = r \Leftrightarrow (u^2 + p)^2 - \frac{q^2}{u^2} = 4r \Leftrightarrow u^6 + 2pu^4 + (p^2 - 4r)u^2 - q^2 = 0$$

Based on the above resolvent cubic equation, solutions for the reduced quartic form are determined as follows:

$$y_{1;2} = \frac{-u \pm \sqrt{u^2 - 4v}}{2} = \frac{1}{2}\left(\left[-u \pm \sqrt{u^2 - 2\left(u^2 + p - \frac{q}{u}\right)}\right]\right) = \frac{1}{2}\left(-u \pm \sqrt{-u^2 - 2p + \frac{2q}{u}}\right)$$

$$y_{3;4} = \frac{u \pm \sqrt{u^2 - 4w}}{2} = \frac{1}{2}\left(\left[u \pm \sqrt{u^2 - 2\left(u^2 + p + \frac{q}{u}\right)}\right]\right) = \frac{1}{2}\left(u \pm \sqrt{-u^2 - 2p - \frac{2q}{u}}\right)$$

And finally, $x = y - \alpha$. This method deals with the quartic equation by solving resolvent cubic and quadratic polynomials, respectively. Therefore, this approach is somehow like a Russian doll, namely Matryoshka.

Considering the quartic equation as obtained in Example 1:

$$x^4 - 3x^3 - 6x^2 + 18x - 9 = 0$$

The Matryoshka approach for $(\alpha; \beta; \gamma; \delta) = \left(-{}^{3}/_{4}; -6; 18; -9\right)$ gives us:

$$\begin{cases} p = -\frac{75}{8} \\ q = \frac{45}{8} \\ r = \frac{45}{256} \end{cases} \Rightarrow \begin{cases} 2p = -\frac{75}{4} \\ p^2 - 4r = \frac{1395}{16} \\ -q^2 = -\frac{2025}{64} \end{cases}$$

Solving the resolvent cubic equation:

$$u^6 - \frac{75}{4}u^4 + \frac{1395}{16}u^2 - \frac{2025}{64} = 0 \Leftrightarrow u^2 \in \left\{\frac{45}{4}; \frac{15}{4} + \frac{3\sqrt{5}}{2}; \frac{15}{4} - \frac{3\sqrt{5}}{2}\right\}$$

Choosing $u^2 = {}^{45}/_{4}$, which implies that $u = {}^{3\sqrt{5}}/_{2}$, solutions for the quartic equation are:

$$x_{4;1} = -\alpha + \frac{1}{2}\left(-u \pm \sqrt{-u^2 - 2p + \frac{2q}{u}}\right) = \frac{1}{4}\left(3 - 3\sqrt{5} \pm \sqrt{30 + 6\sqrt{5}}\right)$$
$$x_{3;2} = -\alpha + \frac{1}{2}\left(u \pm \sqrt{-u^2 - 2p - \frac{2q}{u}}\right) = \frac{1}{4}\left(3 + 3\sqrt{5} \pm \sqrt{30 - 6\sqrt{5}}\right)$$

In Example 1, solving a trigonometricalizable quintic equation is more convenient than solving an order of quartic, cubic, and quadratic equations. Accordingly, we may name this fact "Matryoshka paradox" and further discuss when this paradox happens.

Particularly, we shall find τ (which is not a solution for the generalized quartic polynomial) such that:

$$x^5 - 5\lambda x^3 + 5\lambda^2 x + 2\xi \equiv (x - \tau)(x^4 + 4\alpha x^3 + \beta x^2 + \gamma x + \delta)$$

As both sides are homogeneous quintic polynomials, we get:

$$4\alpha = \tau \qquad \beta = \tau^2 - 5\lambda$$
$$\gamma = \tau\beta = \tau(\tau^2 - 5\lambda) \quad \delta = \tau\gamma + 5\lambda^2 = \tau^2(\tau^2 - 5\lambda) + 5\lambda^2$$

Thus, the Matryoshka paradox happens for a specific class of quartic equations as follow:

$$x^4 + \tau x^3 + (\tau^2 - 5\lambda)x^2 + \tau(\tau^2 - 5\lambda)x + [\tau^2(\tau^2 - 5\lambda) + 5\lambda^2] = 0$$

Exempli gratia, for $\tau = 1$ and $\lambda = 0$, we get:

$$x^4 + x^3 + x^2 + x + 1 = 0 \Leftrightarrow \begin{cases}(x-1)(x^4 + x^3 + x^2 + x + 1) \\ x \neq 1\end{cases} \Leftrightarrow \begin{cases}x^5 = 1 \\ x \neq 1\end{cases}$$

And the solutions are:

$$x \in \left\{exp\left(\frac{\pm 2i\pi}{5}\right); exp\left(\frac{\pm 4i\pi}{5}\right)\right\}$$

Imagination

Considering the following equation:

$$x^5 - 5\lambda x^3 + 5\lambda^2 x + 2\xi = 0 \quad (\lambda, \xi \in \mathbb{C})$$

By choosing one value of $\sqrt{\lambda}$ and replacing $x = 2\sqrt{\lambda}\cos\theta$, we get:

$$\cos 5\theta = -\frac{\xi}{\sqrt{\lambda}} \Leftrightarrow 5\theta = \pm\cos^{-1}\left(-\frac{\xi}{\sqrt{\lambda}}\right) + \mu 2\pi \Leftrightarrow \theta = \pm\frac{1}{5}\cos^{-1}\left(-\frac{\xi}{\sqrt{\lambda}}\right) + \frac{\mu 2\pi}{5} \quad (\mu \in \mathbb{Z})$$

Finally, solutions for the given quintic equation are:

$$x_\nu = 2\sqrt{\lambda}\cos\left[\frac{1}{5}\cos^{-1}\left(-\frac{\xi}{\sqrt{\lambda}}\right) + \frac{\nu 2\pi}{5}\right] \quad (\nu \in \{-2; -1; 0; 1; 2\})$$

Furthermore, the so-called trigonometricalizing method as discussed in this note could be applied to a specific class of odd-ordered algebraic equations.