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SP263. In ΔABC the following relationship holds:

$$12 \le \sum_{cyc}\left(\frac{a+b}{c}\right)^2 \le 18\left(\frac{R}{2r}\right)^2 - 6$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Soumava Chakraborty-Kolkata-India

$$\sum\left(\frac{a+b}{c}\right)^2 - 3 = \sum\left[\left(\frac{b+c}{a}\right)^2 - 1\right] = \sum\left[\frac{(b+c+a)(b+c-a)}{a^2}\right] = \sum\left[\frac{4s(s-a)}{a^2}\right]$$

$$\le \sum\left[\frac{4m_a^2}{a^2}\right] = \sum\left[\frac{2b^2+2c^2-a^2}{a^2}\right]$$

$$= \sum\frac{(2b^2+2c^2+2a^2)-3a^2}{a^2}$$

$$= 2\left(\sum a^2\right)\left(\sum\frac{1}{a^2}\right) - 9 \overset{\text{Leibniz}}{\le} 18R^2\left(\frac{\sum a^2b^2}{16R^2r^2s^2}\right)$$

$$-9 \overset{\text{Goldstone}}{\le} \frac{18(4R^2s^2)}{16r^2s^2} - 9 = 18\left(\frac{R}{2r}\right)^2 - 9$$

$$\Rightarrow \sum\left(\frac{a+b}{c}\right)^2 \overset{(1)}{\le} 18\left(\frac{R}{2r}\right)^2 - 6$$

Again, $\sum\left(\frac{a+b}{c}\right)^2-3=\sum\left[\frac{4s(s-a)}{a^2}\right]\ge 4\sum\frac{w_a^2}{a^2}=4\sum\left[\frac{\left(\frac{4b^2c^2s(s-a)}{bc(b+c)^2}\right)}{a^2}\right]$

$$=4\sum\left[\frac{\left(\frac{bc(b+c+a)(b+c-a)}{(b+c)^2}\right)}{a^2}\right]$$

$$=4\sum\left[\frac{\frac{bc\{(b+c)^2-a^2\}}{(b+c)^2}}{a^2}\right]=4\sum\left[\frac{bc-\frac{a^2bc}{(b+c)^2}}{a^2}\right]$$

$$=4\sum\frac{bc}{a^2}-4\sum\frac{bc}{(b+c)^2}\overset{A-G}{\ge}12\sqrt[3]{\prod\left(\frac{bc}{a^2}\right)}-\sum\frac{4bc}{(b+c)^2}\overset{A-G}{\ge}12-\sum(1)$$

$$=9$$

$\Rightarrow\sum\left(\frac{a+b}{c}\right)^2\overset{(2)}{\ge}12\ \therefore$ combining (1) and (2), $12\le\sum\left(\frac{a+b}{c}\right)^2$

$\le 18\left(\frac{R}{2r}\right)^2-6$ (Proved)