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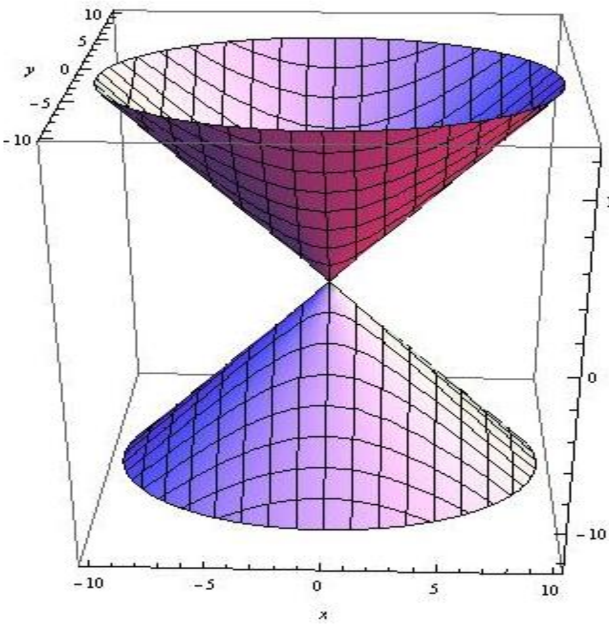

In $\Delta ABC, n_a$ −Nagel's cevian, the following relationship holds:

$$2r \leq \left(\sum_{cyc} \frac{1}{n_a + h_a}\right)^{-1} \leq R$$

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Perpendicular distance is least among all line segments from A to BC, $\therefore h_a$

$$\leq n_a \Rightarrow n_a + h_a \geq 2h_a \Rightarrow \frac{1}{n_a + h_a} \overset{(i)}{\leq} \frac{1}{2h_a}$$

Similarly, $\frac{1}{n_b + h_b} \overset{(ii)}{\leq} \frac{1}{2h_b}$ and $\frac{1}{n_c + h_c} \overset{(iii)}{\leq} \frac{1}{2h_c} \therefore (i) + (ii) + (iii) \Rightarrow \sum \frac{1}{n_a + h_a}$

$$\leq \frac{1}{2}\sum \frac{1}{h_a} = \frac{1}{2}\left(\sum \frac{a}{2rs}\right) = \frac{2s}{4rs} = \frac{1}{2r}$$

$$\Rightarrow \left(\sum \frac{1}{n_a + h_a}\right)^{-1} \overset{(m)}{\geq} 2r$$

Now, Stewart's theorem $\Rightarrow b^2(s - c) + c^2(s - b) = a{n_a}^2 + a(s - b)(s - c)$

$$\Rightarrow s(b^2 + c^2) - bc(2s - a) = a{n_a}^2 + a(s^2 - s(2s - a) + bc)$$

$$\Rightarrow s(b^2 + c^2) - 2sbc = a{n_a}^2 + a(as - s^2)$$

$$\Rightarrow s(b^2+c^2-a^2-2bc)=a{n_a}^2-as^2\Rightarrow a{n_a}^2=as^2+s(2bc\cos A-2bc)$$

$$=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)=as^2-2ah_ar_a\therefore {n_a}^2\overset{(1)}{=}s^2-2h_ar_a$$

Now, $\frac{n_a}{h_a}\le\frac{R}{r}-1\Leftrightarrow\frac{R^2}{r^2}-\frac{2R}{r}+1\ge\frac{n_a^2}{h_a^2}\overset{\text{by (1)}}{\Leftrightarrow}\frac{R^2}{r^2}-\frac{2R}{r}+1\ge\frac{s^2-2h_ar_a}{h_a^2}$

$$=\frac{s^2a^2}{4r^2s^2}-\frac{2r_a}{h_a}=\frac{a^2}{4r^2}-\left(\frac{2rs}{s-a}\right)\left(\frac{a}{2rs}\right)=\frac{a^2}{4r^2}-\frac{(a-s)+s}{s-a}$$

$$=\frac{a^2}{4r^2}+1-\frac{s}{s-a}=1+\frac{a^2(s-a)-4(sr^2)}{4(s-a)r^2}$$

$$=1+\frac{a^2(s-a)-4(s-a)(s-b)(s-c)}{4(s-a)r^2}$$

$$=1+\frac{a^2-(a^2-(b-c)^2)}{4r^2}=1+\frac{(b-c)^2}{4r^2}$$

$$\Leftrightarrow\frac{R^2}{r^2}-\frac{2R}{r}\ge\frac{(b-c)^2}{4r^2}\Leftrightarrow\frac{R(R-2r)}{r^2}\ge\frac{b^2+c^2-2bc}{4r^2}\Leftrightarrow R-2r\ge\frac{b^2+c^2}{4R}-\frac{bc}{2R}$$

$$\Leftrightarrow R\left(1-\frac{2r}{R}\right)\ge\frac{4R^2(\sin^2B+\sin^2C)}{4R}-\frac{4R^2\sin B\sin C}{2R}\Leftrightarrow 1-\frac{8R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R}$$

$$\ge\sin^2B+\sin^2C-2\sin B\sin C=(\sin B-\sin C)^2$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\left(2\sin\frac{B}{2}\sin\frac{C}{2}\right)\ge\left(2\cos\frac{B+C}{2}\sin\frac{B-C}{2}\right)^2$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\left(\cos\frac{B-C}{2}-\cos\frac{B+C}{2}\right)\ge 4\sin^2\frac{A}{2}\left(1-\cos^2\frac{B-C}{2}\right)$$

$$\Leftrightarrow 1-4\sin\frac{A}{2}\cos\frac{B-C}{2}+4\sin^2\frac{A}{2}\ge 4\sin^2\frac{A}{2}-4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2}$$

$$\Leftrightarrow 4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2}-4\sin\frac{A}{2}\cos\frac{B-C}{2}+1\ge 0$$

$$\Leftrightarrow\left(2\sin\frac{A}{2}\cos\frac{B-C}{2}-1\right)^2\ge 0\to\text{true}\Rightarrow\frac{n_a}{h_a}\le\frac{R}{r}-1\Rightarrow\frac{n_a+h_a}{h_a}\le\frac{R}{r}\Rightarrow n_a+h_a$$

$$\le\frac{R}{r}\left(\frac{2rs}{a}\right)=\frac{2Rs}{a}\Rightarrow\frac{1}{n_a+h_a}\overset{(a)}{\ge}\frac{a}{2Rs}$$

Similarly, $\frac{1}{n_b+h_b}\overset{(b)}{\ge}\frac{b}{2Rs}$ and $\frac{1}{n_c+h_c}\overset{(c)}{\ge}\frac{c}{2Rs}\therefore (a)+(b)+(c)\Rightarrow\sum\frac{1}{n_a+h_a}$

$$\ge\frac{2s}{2Rs}=\frac{1}{R}\Rightarrow\left(\sum\frac{1}{n_a+h_a}\right)^{-1}\overset{(n)}{\le}R$$

$$(\mathbf{m}),(\mathbf{n}) \Rightarrow \mathbf{2r} \le \left(\sum \frac{\mathbf{1}}{\mathbf{n_a} + \mathbf{h_a}}\right)^{-\mathbf{1}} \le \mathbf{R}\ (\mathbf{Proved})$$