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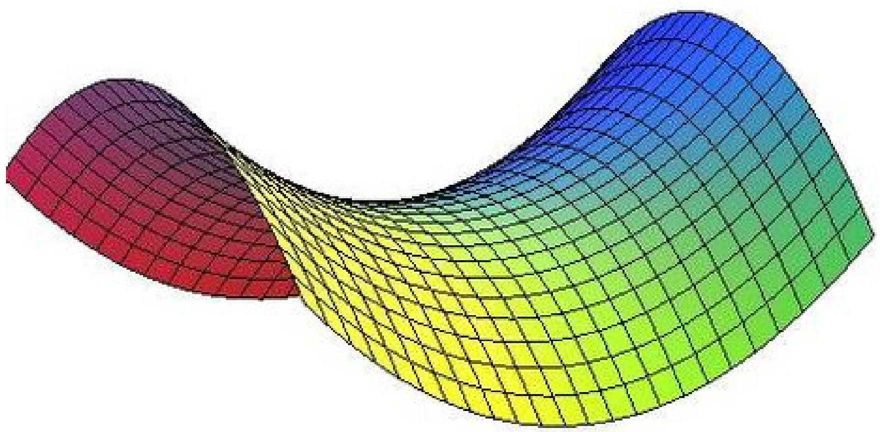

If $a, b, c >, ab + bc + ca = 1$ then:

$$\left(\frac{a^2}{1+a^2}\right)^n + \left(\frac{b^2}{1+b^2}\right)^n + \left(\frac{c^2}{1+c^2}\right)^n \geq \frac{3}{4^n}, n \in \mathbb{N}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Marian Ursărescu-Romania, Solution 2 by George Florin Șerban-Romania, Solution 3 by Michael Sterghiou-Greece

Solution 1 by Marian Ursărescu-Romania

$$\left(\frac{a^2}{1+a^2}\right)^n + \left(\frac{b^1}{1+b^2}\right)^n + \left(\frac{c^2}{1+c^2}\right)^n \geq \frac{\left(\frac{a^2}{1+a^2}+\frac{b^2}{1+b^2}+\frac{c^2}{1+c^2}\right)^n}{3^{n-1}} \Rightarrow \text{we must show:}$$

$$\left(\frac{a^2}{1+a^2}+\frac{b^2}{1+b^2}+\frac{c^2}{1+c^2}\right)^n \geq \frac{3^n}{4^n} \Leftrightarrow \frac{a^2}{1+a^2}+\frac{b^2}{1+b^2}+\frac{c^2}{1+c^2} \geq \frac{3}{4} \quad (1)$$

Because $ab + bc + ca = 1 \Rightarrow \exists x, y, z > 0$ such that:

$$a = \frac{x}{\sqrt{xy+xy+yz}}, b = \frac{y}{\sqrt{xy+xz+yz}}, c = \frac{z}{\sqrt{xy+xz+yz}} \quad (2)$$

From (1)+(2) we must show:

$$\sum \frac{\frac{x^2}{xy+xz+yz}}{1+\frac{x^2}{xy+xz+yz}} \geq \frac{3}{4} \Leftrightarrow \sum \frac{x^2}{xy+xz+yz+x^2} \geq \frac{3}{4}$$

$$\Leftrightarrow \sum \frac{x^2}{(x+y)(x+z)} \geq \frac{3}{4} \quad (3)$$

From Bergström's inequality we have:

$$\sum \frac{x^2}{(x+y)(x+z)} \geq \frac{(x+y+z)^2}{(x+y)(x+z)+(y+x)(y+z)+(z+x)(z+y)} \quad (4)$$

From (3)+(4) we must show:

$$\frac{(x+y+z)^2}{(x+y)(x+z)+(y+x)(y+z)+(z+x)(z+y)} \geq \frac{3}{4} \Leftrightarrow$$

$$\left(2(x+y+z)\right)^2 \geq 3[(x+y)(x+z)+(y+x)(y+z)+(z+y)(z+y)] \Leftrightarrow$$

$$[(x+y)+(x+z)+(y+z)]^2 \geq 3[(x+y)(x+z)+(y+x)(y+z)+(z+y)(z+x)]$$

true because $(\alpha+\beta+\gamma)^2 \geq 3(\alpha\beta+\alpha\gamma+\beta\gamma)$

Solution 2 by George Florin Șerban-Romania

It exists $\Delta ABC, a = \cot\frac{A}{2}, b = \cot B, c = \cot C$

$$\sum_{cyc} ab = \sum_{cyc} \cot A \cdot \cot B = 1$$

$$\frac{a^2}{1+a^2} = \frac{\cot^2 A}{1+\cot^2 A} = \frac{\frac{\cos^2 A}{\sin^2 A}}{1+\frac{\cos^2 A}{\sin^2 A}} = \frac{\cos^2 A}{\sin^2 A} \cdot \frac{\sin^2 A}{1} = \cos^2 A$$

$$\sum_{cyc}\left(\frac{a^2}{1+a^2}\right)^n = \sum_{cyc}(c\cos^2 A)^n \overset{Holder}{\geq} \frac{(\sum\cos^2 A)^n}{3\cdot 3^{n-2}} = \frac{(\sum\cos^2 A)^n}{3^{n-1}}$$

We prove that $\sum\cos^2 A = \frac{6R^2+4Rr+r^2-s^2}{2R^2} \geq \frac{3}{4}$

$$24R^2+16Rr+4r^2-4s^2 \geq 6R^2, 18R^2+16Rr+4r^2 \geq 4s^2$$

$$s^2 \leq \frac{9R^2}{2}+4Rr+r^2, s^2 \overset{Gerretsen}{\leq} 4R^2+4Rr+3r^2 \leq \frac{9R^2}{2}+4Rr+r^2$$

$\Rightarrow 8R^2+8Rr+6r^2 \leq 9R^2+8Rr+2r^2 \Rightarrow R^2 \geq 4r^2 \Rightarrow R \geq 2r$ ***(Euler) (True)***

$\Rightarrow \sum_{cyc}\left(\frac{a^2}{1+a^2}\right)^n \geq \frac{(\sum\cos^2 A)^n}{3^{n-1}} \geq \frac{\left(\frac{3}{4}\right)^n}{3^{n-1}} = \frac{3^n}{4^n\cdot 3^{n-1}} = \frac{3}{4^n}$ ***(True)***

Solution 3 by Michael Sterghiou-Greece

$\sum_{cyc}\left(\frac{a^2}{1+a^2}\right)^n \geq \frac{3}{4^n}$ ***(1)***

Let $(p,q,r) = \left(\sum_{cyc} a, \sum_{cyc} ab, abc\right), q = 1, p^2 \geq 3q \geq 3 \rightarrow p \geq \sqrt{3}$

By convexity of $f(t) = t^n$***, on*** $(0,+\infty)$ ***we have:***

$LHS\ (1) \geq 3 \cdot \left(\frac{\sum_{cyc} \frac{a^2}{1+a^2}}{3}\right)^n \overset{?}{\geq} \frac{3}{4^n}$ ***which reduces to***

$\sum_{cyc} \frac{a^2}{1+a^2} \geq \frac{3}{4}$***. Thus by BCS becomes the still stronger inequality***

$\frac{p^2}{3+\sum_{cyc} a^2} \geq \frac{3}{4}$ ***or*** $\frac{p^2}{p^2+1} \geq \frac{3}{4}$ $\left(p^2 - 2q = \sum_{cyc} a^2\right)$ ***which reduces to*** $p^2 \geq 3$ ***which holds.***

Done!