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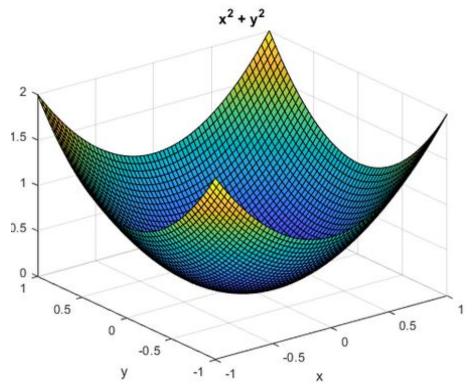

If $a, b, c > 0$ then:

$$\left(\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2}\right)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)^2 \geq \left(\frac{a}{b^2}+\frac{b}{c^2}+\frac{c}{a^2}\right)^3$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania, Solution 2 by Ertan Yildirim-Turkey, Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand, Solution 4 by Tran Hong-Dong Thap-Vietnam

Solution 1 by Adrian Popa-Romania

$$\left(\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2}\right)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)^2 =$$

$$=\left(\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2}\right)\left(\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{a^2}\right)\left(\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{a^2}\right)\overset{J.Holder}{\geq}$$

$$\geq\left(\frac{\sqrt[3]{a^3\cdot 1\cdot 1}}{\sqrt[3]{b^2\cdot b^2\cdot b^2}}+\frac{\sqrt[3]{b^3\cdot 1\cdot 1}}{\sqrt[3]{c^2\cdot c^2\cdot c^2}}+\frac{\sqrt[3]{c^3\cdot 1\cdot 1}}{\sqrt[3]{a^2\cdot a^2\cdot a^2}}\right)^3=\left(\frac{a}{b^2}+\frac{b}{c^2}+\frac{c}{a^2}\right)^3$$

Solution 2 by Ertan Yildirim-Turkey

$$\left(\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2}\right)^{\frac{1}{3}}\cdot\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)^{\frac{1}{3}}\cdot\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)^{\frac{1}{3}}\overset{Holder}{\geq}$$

$$\sqrt[3]{\frac{a^3}{b^2}}\cdot\sqrt[3]{\frac{1}{b^2}}\cdot\sqrt[3]{\frac{1}{b^2}}+\sqrt[3]{\frac{b^3}{c^2}}\cdot\sqrt[3]{\frac{1}{c^2}}\cdot\sqrt[3]{\frac{1}{c^2}}+\sqrt[3]{\frac{c^3}{a^2}}\cdot\sqrt[3]{\frac{1}{a^2}}\cdot\sqrt[3]{\frac{1}{a^2}}$$

$$=\frac{a}{b^2}+\frac{b}{c^2}+\frac{c}{a^2}$$

$$\Rightarrow\left(\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2}\right)\cdot\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\cdot\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\geq\left(\frac{a}{b^2}+\frac{b}{c^2}+\frac{c}{a^2}\right)^3$$

$$\Rightarrow\left(\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2}\right)\cdot\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)^2\geq\left(\frac{a}{b^2}+\frac{b}{c^2}+\frac{c}{a^2}\right)^3$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\textit{For } a,b,c>0 \textit{ we have}$$

$$\left(a^5c^2+c^5b^2+b^5a^2\right)[(ab)^2+(bc)^2+(ca)^2][(ab)^2+(bc)^2+(ca)^2]$$

$$\geq\left\{\left[\left(a^5c^2\right)\left(a^2c^2\right)\left(a^2c^2\right)\right]^{\frac{1}{3}}+\left[\left(c^5b^2\right)\left(c^2b^2\right)\left(c^2b^2\right)\right]^{\frac{1}{3}}+\left[\left(b^5a^2\right)\left(b^2a^2\right)\left(b^2a^2\right)\right]^{\frac{1}{3}}\right\}^3$$

$$=\left[\left(a^9c^6\right)^{\frac{1}{3}}+\left(c^9b^6\right)^{\frac{1}{3}}+\left(b^9a^6\right)^{\frac{1}{3}}\right]^3$$

$$=\left(a^3c^2+c^3b^2+b^3a^2\right)^3$$

$$\textit{Hence}\ \frac{(a^5c^2+c^5b^2+b^5a^2)}{(abc)^2}\frac{[(ab)^2+(bc)^2+(ca)^2]}{(abc)^2}\frac{[(ab)^2+(bc)^2+(ca)^2]}{(abc)^2}$$

$$\geq\left(\frac{a^3c^2+c^3b^2+b^3a^2}{(abc)^2}\right)^3$$

$$\textit{That is}\ \left(\frac{a^3}{b^2}+\frac{c^3}{a^2}+\frac{b^3}{c^2}\right)\left(\frac{1}{c^2}+\frac{1}{a^2}+\frac{1}{b^2}\right)\left(\frac{1}{c^2}+\frac{1}{a^2}+\frac{1}{b^2}\right)$$

$$\geq\left(\frac{a}{b^2}+\frac{c}{a^2}+\frac{b}{c^2}\right)^3$$

Solution 4 by Tran Hong-Dong Thap-Vietnam

$$\left(\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2}\right)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)^2\geq\left(\frac{a}{b^2}+\frac{b}{c^2}+\frac{c}{a^2}\right)^3$$

$$\Leftrightarrow(c^2a^5+a^2b^5+b^2c^5)(a^2b^2+b^2c^2+c^2a^2)^2\geq(c^2a^3+a^2b^3+b^2c^3)^3$$

$$\begin{aligned}\Leftrightarrow a^2b^2c^2\big[&(a^7b^2+b^7c^2+c^7a^2)+2(a^7c^2+a^2b^7+b^2c^7)\\&+2(a^5c^4+a^4b^5+b^4c^5)+2a^2b^2c^2(a^3+b^3+c^3)\\&+a^2b^2c^2(ba^2+ac^2+cb^2)-3(a^5b^4+b^5c^4+c^5a^4)\\&-3abc(ca^5+ab^5+bc^5)-6a^3b^3c^3\big]\geq 0\ (*)\end{aligned}$$

$$\therefore a^7b^2+a^4b^5+a^4b^5\overset{AM-GM}{\geq}3\sqrt[3]{a^{15}b^{12}}=3a^5b^4;(1)$$

$$\because b^7c^2+b^4c^5+b^4c^5\overset{AM-GM}{\geq}3\sqrt[3]{a^{15}b^{12}}=3b^5c^4;(2)$$

$$\because c^7a^2+a^5c^4+a^5c^4\overset{AM-GM}{\geq}3\sqrt[3]{a^{15}b^{12}}=3c^5a^4;(3)$$

$$\overset{(1)+(2)+(3)}{\Longrightarrow} a^7b^2 + b^7c^2 + c^7a^2 + 2\left(a^5c^4 + a^4b^5 + b^4c^5\right) \geq 3\left(a^5b^4 + b^5c^4 + c^5a^4\right) \ (4)$$

$$\because 2a^2b^2c^2(a^3 + b^3 + c^3) \overset{AM-GM}{\geq} 2a^2b^2c^2 \cdot 3 \cdot abc = 6a^3b^3c^3 \ (5)$$

$$\because a^7c^2 + a^7c^2 + (a^2b^2c^2 \cdot ba^2) \overset{AM-GM}{\geq} 3\sqrt[3]{a^{18}b^3c^6} = 3abc\left(ca^5\right) \ (6)$$

$$\because a^2b^7 + a^2b^7 + (a^2b^2c^2 \cdot cb^2) \overset{AM-GM}{\geq} 3\sqrt[3]{b^{18}a^6c^3} = 3abc\left(ab^5\right) \ (7)$$

$$\because b^2c^7 + b^2c^7 + (a^2b^2c^2 \cdot ac^2) \overset{AM-GM}{\geq} 3\sqrt[3]{c^{18}b^6a^3} = 3abc\left(bc^5\right) \ (8)$$

$$\overset{(6)+(7)+(8)}{\Longrightarrow} 2(a^7c^2 + c^7b^2 + b^7a^2) + a^2b^2c^2(ac^2 + cb^2 + ba^2) \geq 3abc\left(ab^5 + bc^5 + ca^5\right) \ (9)$$

$$\overset{(4),(5),(9),a,b,c\geq 0}{\Longrightarrow} (*) \ \textit{is true}.$$

Proved.