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If $a, b, c > 0, a^2 + b^2 + c^2 + 2abc = 1$ then:

$$8(1-a)(1-b)(1-c) \le 1$$

Proposed by Marin Chirciu-Romania

Solution 1 by Marian Ursărescu-Romania, Solution 2 by Michael Sterghiou-Greece, Solution 3 by Rajeev Rastogi-India

Solution 1 by Marian Ursărescu-Romania

Because we have the next relation:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma = 1, \forall \alpha, \beta, \gamma \in (0, \pi)$$

with $\alpha + \beta + \gamma = \pi \Rightarrow \exists \Delta ABC$ *with*

$A, B, C \in (0, \pi)$ *and* $A + B + C = \pi$ *and* $a = \cos A, b = \cos B, c = \cos C$

$\Rightarrow$ *we must show:* $8(1-\cos A)(1-\cos B)(1-\cos C) \le 1 \Leftrightarrow$

$$64 \cdot \sin^2\frac{A}{2} \cdot \sin^2\frac{B}{2} \cdot \sin^2\frac{C}{2} \le 1 \Leftrightarrow \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{1}{8} \quad (1)$$

But $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \le \frac{a}{2\sqrt{bc}}$ *and similarly* $\Rightarrow$

$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{a}{2\sqrt{bc}} \cdot \frac{b}{2\sqrt{ac}} \cdot \frac{c}{2\sqrt{ab}} = \frac{1}{8} \Rightarrow$ *(1) it is true.*

Solution 2 by Michael Sterghiou-Greece

$a^2 + b^2 + c^2 + 2abc = 1$ *(c) then* $8(1-a)(1-b)(1-c) \le 1$ *(1)*

From (c) $1=\sum_{cyc}a^2+2abc\overset{AM-GM}{\geq}3r^{\frac{2}{3}}+2r$ *(2)* $(r=:abc)\to r\leq\frac{1}{8}$

[if $r=t^3$ *(2)*$\to 3t^2+2t^3-1\leq 0\to(t+1)^2(2t-1)\leq 0\to$

$t\leq\frac{1}{2}\to r\leq\frac{1}{8}$*]. Now, let* $p=\sum_{cyc}a, a-\sum_{cyc}ab$*. If* $\max\{a,b,c\}<\frac{1}{2}$ *then form (c)*

$\sum a^2+2r<1$ *hence* $\max\{a,b,c\}\geq\frac{1}{2}$ *and* $p\geq\frac{1}{2}$*.*

Assume $p\geq\frac{3}{2}\to -p\leq-\frac{3}{2}\to 3-p\leq\frac{3}{2}\to\frac{2}{3}\cdot\sum_{cyc}(1-a)\leq 1$

$$\to\frac{2^3}{3^3}\left[\sum_{cyc}(1-a)\right]^3\leq 1\to 3^3\cdot\prod_{cyc}(1-a)\cdot\frac{2^3}{3^3}\leq 1\to 8\prod_{cyc}(1-a)\leq 1$$

Done.

Assume now $p\leq\frac{3}{2}$*. LHS of (1)* $\to 8(q-p-r+1)$ *so (1)*

reduces to: $8(p-q+r)-7\geq 0$*. With* $r=\frac{1+2q-p^2}{2}$ *from (c)*

the last inequality reduces to $-4p^2+8p-3\geq 0$ *or*

$(2p-1)(3-2p)\geq 0$ *which holds as* $p\geq\frac{1}{2}$ *and* $p\leq\frac{3}{2}$*.*

We are done!

Solution 3 by Rajeev Rastogi-India

Let us try with change of variables $(\because 0<a,b,c<1)$

$a=\cos A$, $b=\cos B$, $c=\cos C$ *where* $A,B,C\in\left(0,\frac{\pi}{2}\right)$

$\therefore$ *Given condition becomes* $\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1$

$$\Rightarrow\cos^2 A+(\cos^2 B-\sin^2 C)+\cos A\,(2\cos B\cos C)=0$$

$$\Rightarrow\cos^2 A+\cos(B+C)\cdot\cos(B-C)+\cos A\,(\cos(B+C)+\cos(B-C))=0$$

$$\Rightarrow(\cos^2 A+\cos A\cos(B+C))+(\cos(B+C)\cdot\cos(B-C)+\cos A\cos(B-C))=0$$

$$\Rightarrow\cos(\cos A+\cos(B+C))+\cos(B-C)\,(\cos A+\cos(B+C))=0$$

$$\Rightarrow(\cos A+\cos(B+C))\cdot\underbrace{(\cos A+\cos(B-C))}_{\neq(as\ A,B,C\ are\ acute\ angles)}=0$$

$$\Rightarrow\cos(B+C)=-\cos A=\cos(\pi-A)$$

$$\Rightarrow A+B+C=\pi$$

Let $E=(1-a)(1-b)(1-c)=8\sin^2\frac{A}{2}\cdot\sin^2\frac{B}{2}\cdot\sin^2\frac{C}{2}$ *(1)*

As the function $\ln\sin\frac{x}{2}$ ***is concave down so by Jensen's inequality***

$$f\left(\frac{A+B+C}{3}\right) \geq \frac{f(A)+f(B)+f(C)}{3} \textit{ where } f(x) = \ln\sin\frac{x}{2}$$

$$\Rightarrow \sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2} \leq \frac{1}{8}$$

$$\therefore (1) \Rightarrow E \leq 8\cdot\left(\frac{1}{8}\right)^2 \Rightarrow E \leq \frac{1}{8}$$