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If $a,b>0, a\neq b$ then:

$$\frac{1}{\frac{1}{2a+b}+\frac{1}{a+2b}+\frac{(a+b)^2}{12}-1}+\frac{1}{\frac{1}{2a+b}+\frac{1}{a+2b}+\frac{(a+b)^2}{12}+1}>\frac{1}{\frac{1}{2a+b}+\frac{1}{a+2b}+\frac{(a+b)^2}{12}}$$

Proposed by Petre Rau-Romania

Solution 1 by Florentin Vișescu-Romania, Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand, Solution 3 by Soumava Chakraborty-Kolkata-India

Solution 1 by Florentin Vișescu-Romania

We prove that $\frac{1}{2a+b}+\frac{1}{2b+a}+\frac{(a+b)^2}{12}>1; \forall a,b\in(0,\infty); a\neq b$

$$\frac{2}{\frac{1}{2a+b}+\frac{1}{2b+a}}<\frac{2a+b+2b+a}{2}\Leftrightarrow (\text{“}<\text{”} \because 2a+b\neq 2b+a; a\neq b)$$

$$\frac{1}{2a+b}+\frac{1}{2b+a}>\frac{4}{3(a+b)}\Bigg|\frac{(a+b)^2}{12}\Rightarrow$$

$$\frac{1}{2a+b}+\frac{1}{2b+a}+\frac{(a+b)^2}{12}>\frac{4}{3(a+b)}+\frac{(a+b)^2}{12}$$

We prove that $\frac{4}{3(a+s)}+\frac{(a+b)^2}{12}\geq 1; \frac{4}{3s}+\frac{s^2}{12}\geq 1$

$$16+s^3\geq 12s; s^3-12s+16\geq 0$$

	s^3	s^2	s	s^0
	1	0	12	-16

2	1	2	−8	0
2	1	4	0	

$(s-2)^2(s+4) \geq 0, \forall s > 0$ ***(true)***

So, $\underbrace{\frac{1}{2a+s}+\frac{1}{2b+a}+\frac{(a+b)^2}{12}}_{t} > 1$

We prove that for $t > 1$

$\frac{1}{t-1}+\frac{1}{t+1} > \frac{1}{t}$ ***or*** $t(t+1)+t(t-1) > t^2-t^2+t+t^2-t > t^2-1$

$t^2 > -1$ ***(True)***

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a, b > 0$ ***and*** $a \neq b$, ***we consider that***

$\frac{1}{2a+b}+\frac{1}{a+2b}+\frac{(a+b)^2}{12} > 1$ ***as follows***

give $2a+b = x$ ***and*** $a+2b = y \Rightarrow \frac{x+y}{3} = a+b$

Hence $\frac{1}{2a+b}+\frac{1}{a+2b}+\frac{(a+b)^2}{12} = \frac{1}{x}+\frac{1}{y}+\frac{(x+y)^2}{108} : a \neq b \leftrightarrow x \neq y$

$$> \frac{1}{x}+\frac{1}{y}+\frac{4xy}{108} = \frac{1}{x}+\frac{1}{y}+\frac{xy}{27}$$

$> 1 : a \neq b \leftrightarrow x \neq y$

Hence $\frac{1}{\frac{1}{2a+b}+\frac{1}{a+2b}+\frac{(a+b)^2}{12}-1}+\frac{1}{\frac{1}{2a+b}+\frac{1}{a+2b}+\frac{(a+b)^2}{12}+1} >$

$$\geq \frac{4}{\frac{2}{2a+b}+\frac{2}{a+2b}+\frac{2(a+b)^2}{12}}$$

$= \frac{2}{\frac{1}{2a+b}+\frac{1}{a+2b}+\frac{(a+b)^2}{12}} > \frac{1}{\frac{1}{2a+b}+\frac{1}{a+2b}+\frac{(a+b)^2}{12}}$ ***OK***

Therefore it is true.

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\frac{1}{\frac{1}{2a+b}+\frac{1}{a+2b}+\frac{(a+b)^2}{12}-1}+\frac{1}{\frac{1}{2a+b}+\frac{1}{a+2b}+\frac{(a+b)^2}{12}+1} >$$

$$\overset{(1)}{\Leftarrow}\ \frac{1}{\frac{1}{2a+b}+\frac{1}{a+2b}+\frac{(a+b)^2}{12}}$$

Let $\frac{1}{2a+b}+\frac{1}{a+2b}=x,\ \frac{(a+b)^2}{12}=y$

Then (1) $\Leftrightarrow \frac{1}{x+y-1}+\frac{1}{x+y+1}\overset{(2)}{>}\frac{1}{x+y}$

Now, $x+y=\frac{1}{2a+b}+\frac{1}{a+2b}+y$

$$\overset{Bergstrom}{>}\ \frac{4}{3(a+b)}+\frac{(a+b)^2}{12}$$

$$\left(\because \frac{1}{2a+b}\neq\frac{1}{a+2b}\ as\ a\neq b\right)$$

$$=\frac{2}{3(a+b)}+\frac{2}{3(a+b)}+\frac{(a+b)^2}{12}$$

$$\overset{A-G}{\geq}\ 3\sqrt[3]{\frac{4(a+b)^2}{108(a+b)^2}}=1\Rightarrow x+y-1>0$$

$\therefore$ **LHS of (2)** $\overset{Bergstrom}{>}\ \frac{4}{2x+2y}$

$$\left(\because \frac{1}{x+y-1}\neq\frac{1}{x+y+1}\right)$$

$=\frac{2}{x+y}>\frac{1}{x+y}\Rightarrow$ **(2)** $\Rightarrow$ **(1) is true. Observation: LHS** >2 **(LHS) actually.**