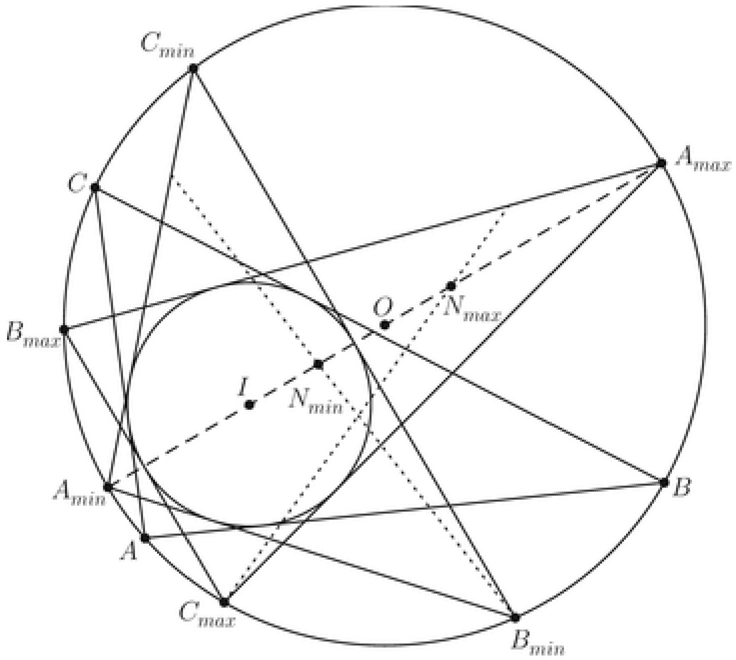
R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

A NEW DEMONSTRATION OF RAMUS-E.ROUCHE-BLUNDON DOUBLE INEQUALITY AND IT'S CONSEQUENCES

By Marian Dincă – Romania



THEOREM:

Let ABC be a triangle that has its circumcircle $C(0, R)$ and the incircle $C(I, r)$ Then, it exists two isosceles triangles inscribed to the circle $C(O, R)$ and circumscribed to the circle $C(I, r)$

Proof:

Without using Poncelet theorem we use Carnot identity:

$$\cos A + \cos B + \cos C = \frac{R + r}{R}$$

We will prove that there exists two angles α respectively β such that:

$$\cos A + \cos B + \cos C = \cos \alpha + 2 \cos \left(\frac{\pi - \alpha}{2}\right) = \cos \beta + 2 \cos \left(\frac{\pi - \beta}{2}\right)$$

Such that: $\beta \geq A \geq B \geq C \geq \alpha$

Elementary proof:

$$\cos A+\cos B+\cos C=\cos x+2\cos\left(\frac{\pi-x}{2}\right)=1-2\sin^2\left(\frac{x}{2}\right)+2\sin\left(\frac{x}{2}\right)$$

$$\text{or: } 2\sin^2\left(\frac{x}{2}\right)-2\sin\left(\frac{x}{2}\right)+\cos A+\cos B+\cos C-1=0$$

$$\sin\left(\frac{x}{2}\right)=\frac{1\pm\sqrt{1-2(\cos A+\cos B+\cos C-1)}}{2}=$$

$$=\frac{1\pm\sqrt{3-2(\cos A+\cos B+\cos C)}}{2}=\frac{1\pm\sqrt{3-2\left(\frac{R+r}{R}\right)}}{2}$$

$$=\frac{1\pm\sqrt{\frac{R-2r}{R}}}{2}$$

So: $\sin\frac{x_1}{2}\le\frac{1}{2}$, it follows: $\frac{x_1}{2}\le\frac{\pi}{6}$, let be $x_1=\alpha\le\frac{\pi}{3}$

respectively: $\sin\frac{x_2}{2}\ge\frac{1}{2}$, let be $x_2=\beta\ge\frac{\pi}{3}$

Let be: $A\ge B\ge C$,

$$\cos A+\cos B+\cos C=\cos\alpha+2\cos\left(\frac{\pi-a}{2}\right)$$

but: $\cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\le 2\cos\left(\frac{A+B}{2}\right)$

It follows: $\cos A+\cos B+\cos C\le 2\cos\left(\frac{A+B}{2}\right)+\cos C=\cos C+2\cos\left(\frac{\pi-C}{2}\right)$

So: $\cos\alpha+2\cos\left(\frac{\pi-a}{2}\right)\le\cos C+2\cos\left(\frac{\pi-C}{2}\right)$. The function:

$f(x)=\cos x+2\cos\left(\frac{\pi-x}{2}\right), x\in\left[0,\frac{\pi}{3}\right]$, $f'(x)=-\sin x+\sin\left(\frac{\pi-x}{2}\right)\ge 0$, so, increasing

$f(\alpha)\le f(C)$, implies: $\alpha\le C$. Also: $\cos B+\cos C\le 2\cos\left(\frac{B+C}{2}\right)=2\cos\left(\frac{\pi-A}{2}\right)$

$$\cos A+\cos B+\cos C\le\cos A+2\cos\left(\frac{\pi-A}{2}\right)$$

But: $\cos\beta+2\cos\left(\frac{\pi-B}{2}\right)=\cos A+\cos B+\cos C\le\cos A+2\cos\left(\frac{\pi-A}{2}\right)$

So: $\cos\beta+2\cos\left(\frac{\pi-B}{2}\right)\le\cos A+2\cos\left(\frac{\pi-A}{2}\right)$

The function: $f(x)=\cos x+2\cos\left(\frac{\pi-x}{2}\right), x\in\left[\frac{\pi}{3},\pi\right]$

$f'(x)=-\sin x+\sin\left(\frac{\pi-x}{2}\right)\le 0$, decreasing, so: $A\le\beta$

In the end there are two isoscel triangles that satisfy Carnot relationship:

$$\cos A+\cos B+\cos C=\cos\alpha+2\cos\left(\frac{\pi-\alpha}{2}\right)=\cos\beta+2\cos\left(\frac{\pi-\beta}{2}\right)=\frac{R+r}{R}$$

And: $\beta\geq A\geq B\geq C\geq\alpha$

Let $A_1B_1C_1$ be an isoscelles triangle with:

$$B_1C_1=2R\sin\alpha=2r\cdot\cot\left(\frac{\pi-\alpha}{4}\right), A_1B_1=A_1C_1=2R\sin\left(\frac{\pi-\alpha}{2}\right)$$

$$=r\cdot\cot\frac{\alpha}{2}+r\cdot\cot\left(\frac{\pi-\alpha}{4}\right)$$

we prove that: $AB+BC+CA\leq A_1B_1+B_1C_1+C_1A_1$ (i)

$$AB+BC+CA=2R\cdot(\sin A+\sin B+\sin C)=2r\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)$$

Or: $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\leq\cot\frac{\alpha}{2}+2\cot\left(\frac{\pi-\alpha}{4}\right)$ (ii)

$A+B-2\left(\frac{\pi-\alpha}{2}\right)=\alpha-C\leq 0$, it follows: $A+B-2\left(\frac{\pi-\alpha}{2}\right)=\alpha-C\leq 0$, it follows:

$A+B-2\left(\frac{\pi-\alpha}{2}\right)\leq 0, A+B\leq 2\left(\frac{\pi-\alpha}{2}\right)$. So: $B\leq\frac{A+B}{2}\leq\frac{\pi-\alpha}{2}$, it follows:

$\alpha\leq C\leq B\leq\frac{\pi-\alpha}{2}$, then it exists α and $\mu\in[0,1]$ such that:

$$C=\lambda\cdot\left(\frac{\pi-\alpha}{2}\right)+(1-\lambda)\cdot\alpha\quad (1)$$

$$B=\mu\cdot\left(\frac{\pi-\alpha}{2}\right)+(1-\mu)\cdot\alpha\quad (1')$$

$$B\geq C, \text{ implies: } \mu\cdot\left(\frac{\pi-\alpha}{2}\right)+(1-\mu)\cdot\alpha\geq\lambda\cdot\left(\frac{\pi-\alpha}{2}\right)+(1-\alpha)\cdot\alpha$$

$$\text{Or: } (\mu-\alpha)\cdot\left(\frac{\pi-\alpha}{2}\right)-(\mu-\lambda)\cdot\alpha\geq 0, \text{ or } (\mu-\lambda)\left(\frac{\pi-\alpha}{2}-\alpha\right)\geq 0$$

It follows: $\mu\geq\lambda$

$$B+C\leq\frac{2\pi}{3}, (\alpha+\mu)\cdot\left(\frac{\pi-\alpha}{2}\right)+(2-\lambda-\mu)\cdot\alpha\leq\frac{2}{3}\cdot\pi=\frac{2}{3}\left[2\cdot\left(\frac{\pi-\alpha}{2}\right)+\alpha\right]$$

$$\text{Or: }\left(\lambda+\mu-\frac{4}{3}\right)\left(\frac{\pi-\alpha}{2}\right)+\left(2-\lambda-\mu-\frac{2}{3}\right)\cdot\alpha\leq 0$$

$$\text{Or: }\left(\alpha+\mu-\frac{4}{3}\right)\left(\frac{\pi-\alpha}{2}-\alpha\right)\leq 0, \text{ it follows: } \alpha+\mu-\frac{4}{3}\leq 0$$

Inequality (ii) is equivalent with:

$$\cot\left(\frac{\lambda\left(\frac{\pi-\alpha}{2}\right)+(1-\lambda)\cdot\alpha}{2}\right)+\cot\left(\frac{\mu\left(\frac{\pi-\alpha}{2}\right)+(1-\mu)\cdot\alpha}{2}\right)+$$

R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

$$+\tan\left(\frac{\lambda\left(\frac{\pi-\alpha}{2}\right)+(1-\lambda)\cdot\alpha}{2}+\frac{\mu\left(\frac{\pi-\alpha}{2}\right)+(1-\mu)\cdot\alpha}{2}\right)=$$

$$=\cot\left(\left(\frac{\pi-\alpha}{4}\right)\cdot\alpha+\frac{\alpha}{2}(1-\lambda)\right)+\cot\left(\left(\frac{\pi-\alpha}{4}\right)\cdot\mu+\frac{\alpha}{2}\cdot(1-\mu)\right)+$$

$$+\tan\left(\left(\frac{\pi-\alpha}{2}\right)\cdot\left(\frac{\lambda+\mu}{2}\right)+\alpha\left(1-\frac{\lambda+\mu}{2}\right)\right)=$$

$$=\cot\left(\left(\frac{\pi-\alpha}{4}\right)\cdot\lambda+\frac{\alpha}{2}\cdot(1-\lambda)\right)+\cot\left(\left(\frac{\pi-\alpha}{4}\right)\cdot\mu+\frac{\alpha}{2}\cdot(1-\mu)\right)+$$

$$+\tan\left(\left(\frac{\pi-\alpha}{2}\right)\cdot\left(\frac{\lambda+\mu}{2}\right)+\alpha\cdot\left(1-\frac{\alpha+\mu}{2}\right)\right)=$$

$$=\cot\left[\lambda\cdot\left(\frac{\pi-3\alpha}{4}\right)+\frac{\alpha}{2}\right]+\cot\left[\mu\left(\frac{\pi-3\alpha}{4}\right)+\frac{\alpha}{2}\right]+\tan\left[\left(\frac{\alpha+\mu}{2}\right)\cdot\left(\frac{\pi-3\alpha}{2}\right)+\alpha\right]$$

Let be $f:[0,1]\times[0,1]\to\mathbb{R}$

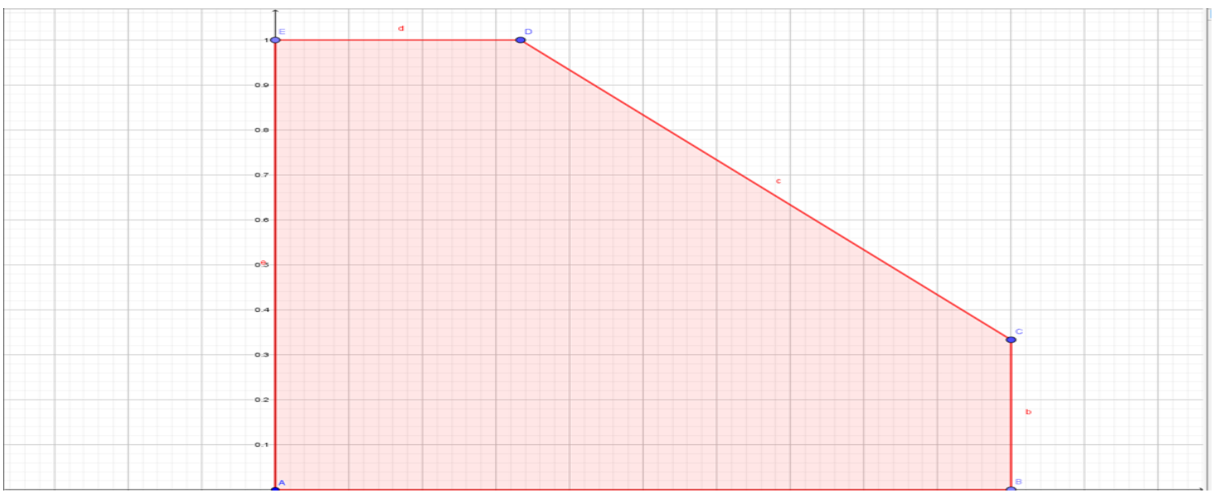
$$f(\lambda,\mu)=\cot\left[\lambda\cdot\left(\frac{\pi-3\alpha}{4}\right)+\frac{\alpha}{2}\right]+\cot\left[\mu\left(\frac{\pi-3\alpha}{4}\right)+\frac{\alpha}{2}\right]+\tan\left[\left(\frac{\lambda+\mu}{2}\right)\cdot\left(\frac{\pi-3\alpha}{2}\right)+\alpha\right]$$

is a convex function defined on a convex set: $[0,1]\times[0,1]\cap\left\{(\lambda,\mu)|\lambda+\mu\le\frac{4}{3}\right\}$

So the pentagon $ABCDE$, where: $A(0,0), B(1,0), C\left(1,\frac{1}{3}\right), D\left(\frac{1}{3},1\right), E(0,1)$

Will attain its maximum in one of these peaks.

Any convex function defined on a convex set will attain its maximum in its extreme points.



$$f(1,0)=\cot\left(\frac{\pi-\alpha}{4}\right)+\cot\frac{\alpha}{2}+\tan\left(\frac{\pi+\alpha}{4}\right)=\cot\left(\frac{\pi-\alpha}{4}\right)+\cot\frac{\alpha}{2}+\cot\left(\frac{\pi}{2}-\frac{\pi+\alpha}{4}\right)=$$

$$=\cot\frac{\alpha}{2}+2\cot\left(\frac{\pi-\alpha}{4}\right)$$

$$f\left(1,\frac{1}{3}\right)=\cot\left(\frac{\pi-\alpha}{4}\right)+\cot\left(\frac{1}{3}\cdot\left(\frac{\pi-3\alpha}{4}\right)+\frac{\alpha}{2}\right)+\tan\left(\frac{\pi-3\alpha}{3}+\alpha\right)=$$

$$=\cot\left(\frac{\pi-\alpha}{4}\right)+\cot\left(\frac{\pi+3\alpha}{12}\right)+\tan\frac{\pi}{3}\le\cot\frac{\alpha}{2}+2\cot\left(\frac{\pi-\alpha}{4}\right),\text{ or:}$$

$$\cot\frac{\alpha}{2}-\cot\left(\frac{\pi+3\alpha}{12}\right)+\cot\left(\frac{\pi-\alpha}{4}\right)\ge\tan\frac{\pi}{3}=\cot\frac{\pi}{6}$$

$\frac{\alpha}{2}\le\frac{\pi+3\alpha}{12}\le\frac{\pi-3\alpha}{4}$, it follows: $\frac{\pi+3\alpha}{12}=\gamma\cdot\frac{\pi-\alpha}{4}+(1-\gamma)\cdot\frac{\alpha}{2}=$

$$=\gamma\cdot\left(\frac{\pi-\alpha}{4}-\frac{\alpha}{2}\right)+\frac{\alpha}{2},\gamma\in[0,1]$$

$$\cot\left(\frac{\pi+3\alpha}{12}\right)=\cot\left(\gamma\cdot\frac{\pi-\alpha}{4}+(1-\gamma)\cdot\frac{\alpha}{2}\right)\le\gamma\cdot\cot\left(\frac{\pi-\alpha}{4}\right)+(1-\gamma)\cot\frac{\alpha}{2}$$

So: $-\cot\left(\frac{\pi+3\alpha}{12}\right)\ge-\gamma\cdot\cot\left(\frac{\pi-\alpha}{4}\right)-(1-\gamma)\cot\frac{\alpha}{2}$

We obtain: $\cot\frac{\alpha}{2}-\cot\left(\frac{\pi+3\alpha}{12}\right)+\cot\left(\frac{\pi-\alpha}{4}\right)\ge\cot\frac{\alpha}{2}-\gamma\cdot\cot\left(\frac{\pi-\alpha}{4}\right)-(1-\gamma)\cot\frac{\alpha}{2}+\cot\left(\frac{\pi-\alpha}{4}\right)=$

$$=\gamma\cot\frac{\alpha}{2}+(1-\gamma)\cot\left(\frac{\pi-\alpha}{4}\right)\ge\cot\left(\gamma+(1-\gamma)\right)\left(\frac{\pi-\alpha}{4}\right)$$

$$=\cot\left[\frac{\pi-\alpha}{4}-\gamma\cdot\left(\left(\frac{\pi-\alpha}{4}-\frac{\alpha}{2}\right)\right)\right]=$$

$$=\cot\left[\frac{\pi-\alpha}{4}+\frac{\alpha}{2}-\frac{\pi+3\alpha}{12}\right]=\cot\left(\frac{3\pi-3\alpha+6\alpha-\pi-3\alpha}{12}\right)=\cot\frac{\pi}{6}$$

But: $f(1,0)=f(0,1)$ and $f\left(1,\frac{1}{3}\right)=f\left(\frac{1}{3},1\right)$ being symmetric.

$B+C-2\left(\frac{\pi-\beta}{2}\right)=\beta-A\ge0$, it follows: $B+C\ge2\left(\frac{\pi-\beta}{2}\right)$

We prove the other inequality:

$AB+BC+CA\ge A_1B_2+B_2C_2+C_2A_2$, or so: $B\ge\frac{B+C}{2}\ge\frac{\pi-\beta}{2}$, it follows: $B\ge\frac{\pi-\beta}{2}$

$\sin A+\sin B+\sin C\ge\sin\beta+2\sin\left(\frac{\pi-\beta}{2}\right)$ we obtain: $\beta\ge A\ge B\ge\frac{\pi-\beta}{2}$

Then it exists λ respectively $\mu\in[0,1]$, such that:

$$A=\lambda\cdot\beta+(1-\lambda)\cdot\left(\frac{\pi-\beta}{2}\right)$$

$$B=\mu\cdot\beta+(1-\mu)\cdot\left(\frac{\pi-\beta}{2}\right)$$

$$A-B=(\alpha-\mu)\cdot\beta-(\alpha-\mu)\left(\frac{\pi-\beta}{2}\right)=(\alpha-\mu)\left(\beta-\frac{\pi-\beta}{2}\right)\geq 0$$

It follows: $\lambda\geq\mu$, because $\beta-\frac{\pi-\beta}{2}\geq 0$

$$A+B\geq\frac{2\pi}{3}$$

$$\alpha\cdot\beta+(1-\alpha)\cdot\left(\frac{\pi-\beta}{2}\right)+\mu\cdot\beta+(1-\mu)\cdot\left(\frac{\pi-\beta}{2}\right)\geq\frac{2\pi}{3}$$

$$(\alpha-\mu)\cdot\beta+(2-\lambda-\mu)\cdot\left(\frac{\pi-\beta}{2}\right)\geq\frac{2}{3}\cdot\pi=\frac{2}{3}\left[\beta+2\left(\frac{\pi-\beta}{2}\right)\right]$$

It follows: $\left(\lambda+\mu-\frac{2}{3}\right)\cdot\beta+\left(2-\lambda-\mu-\frac{4}{3}\right)\left(\frac{\pi-\beta}{2}\right)\geq 0$

Or: $\left(\alpha+\mu-\frac{2}{3}\right)\cdot\left(\beta-\frac{\pi-\beta}{2}\right)\geq 0$

It follows: $\lambda+\mu-\frac{2}{3}\geq 0$

$$\sin A+\sin B+\sin C=$$

$$=\sin\left(\alpha\beta+(1-\lambda)\left(\frac{\pi-\beta}{2}\right)\right)+\sin\left(\mu\beta+(1-\mu)\left(\frac{\pi-\beta}{2}\right)\right)+$$

$$+\sin\left[\alpha\beta+(1-\lambda)\left(\frac{\pi-\beta}{2}\right)+\mu\beta+(1-\mu)\left(\frac{\pi-\beta}{2}\right)\right]=$$

$$=\sin\left[\lambda\left(\beta-\frac{\pi-\beta}{2}\right)+\frac{\pi-\beta}{2}\right]+\sin\left[\mu\left(\beta-\frac{\pi-\beta}{2}\right)+\frac{\pi-\beta}{2}\right]+$$

$$+\sin\left[(\lambda+\mu)\left(\beta-\frac{\pi-\beta}{2}\right)+\pi-\beta\right]=g(\lambda,\mu)$$

$$g:[0,1]\times[0,1]\to\mathbb{R}$$

g is concave defined on the convex set $[0,1]\times[0,1]\cap\left\{(\lambda,\mu)|\lambda+\mu\geq\frac{2}{3}\right\}$

It will attain its maximum in the points $A\left(\frac{2}{3},0\right),B(1,0),C(1,1),D(0,1),F\left(0,\frac{2}{3}\right)$

Being simetric it remains to verify $A\left(\frac{2}{3},0\right),B(1,0),C(1,1)$

$$g\left(\frac{2}{3},0\right)=\sin\left(\frac{2}{3}\cdot\left(\beta-\frac{\pi-\beta}{2}\right)+\frac{\pi-\beta}{2}\right)+\sin\left(\frac{\pi-\beta}{2}\right)+\sin\left(\frac{2}{3}\cdot\left(\beta-\frac{\pi-\beta}{2}\right)+\pi-\beta\right)=$$

$$=\sin\left(\frac{3\beta+\pi}{6}\right)+\sin\left(\frac{\pi-\beta}{2}\right)+\sin\frac{2\pi}{3}\geq\sin\beta+2\sin\left(\frac{\pi-\beta}{2}\right)$$

Or: $\sin\frac{2\pi}{3}\geq\sin\beta-\sin\left(\frac{3\beta+\pi}{6}\right)+\sin\left(\frac{\pi-\beta}{2}\right)$

Using the same technique: $\beta\geq\frac{3\beta+\pi}{6}\geq\frac{\pi-\beta}{2}$

Or Steffensen in discreete form

$$\sin\beta-\sin\left(\frac{3\beta+\pi}{6}\right)+\sin\left(\frac{\pi-\beta}{2}\right)\leq\sin\left(\beta-\frac{3\beta+\pi}{6}+\frac{\pi-\beta}{2}\right)=$$

$$=\sin\left(\frac{6\beta-3\beta-\pi+3\pi-3\beta}{6}\right)=\sin\frac{\pi}{3}$$

Ramus-E, Rouche, Blundon for any acute angled triangle:

If ABC triangle is acute-angled, that has the circumcircle triangle $C(0,R)$, and the inscribed circle $C(I,r)$ and $A,B,C\in\left[0,\frac{\pi}{2}\right]$

Then it exists two acute-angled triangles that has the same R respectively r and it exists the double inequality:

$s_1\leq s\leq s_2$ where s_1,s_2,s are semiperimeters of the those triangles.

Proof:

In the first part of the article we've proved that:

$$\alpha\leq C\leq B\leq A\leq\beta$$

In the situation that the triangle is acute-angled, we can replace

$\{A,B,C\}$ with $\left\{\frac{\pi-A}{2},\frac{\pi-B}{2},\frac{\pi-C}{2}\right\}$

the triangle that has the angles having the measure: $\frac{\pi-A}{2},\frac{\pi-\beta}{2},\frac{\pi-C}{2}$, is acute angled and:

$\frac{\pi-A}{2}+\frac{\pi-B}{2}+\frac{\pi-C}{2}=\pi$, and

$$\frac{\pi-\beta}{2}\leq\frac{\pi-A}{2}\leq\frac{\pi-B}{2}\leq\frac{\pi-C}{2}\leq\frac{\pi-\alpha}{2}$$

In the same way like in the first part it will follow:

$$\sin\left(\frac{\pi-\alpha}{2}\right)+2\sin\left(\frac{\pi-\frac{\pi-\alpha}{2}}{2}\right)\le\sin\left(\frac{\pi-A}{2}\right)+\sin\left(\frac{\pi-B}{2}\right)+\sin\left(\frac{\pi-C}{2}\right)\le$$

$$\le\sin\left(\frac{\pi-\beta}{2}\right)+2\sin\left(\frac{\pi-\frac{\pi-\beta}{2}}{2}\right)$$

So: $s_1\le s\le s_2$

Bibliography:

[1] R. Tyrrel Rockafellar: Convex Analysis, Theta Publishing House, 2002.

[2] Arthur Engel: Mathematical problems, Solving strategies, GIL Publishing House, 2006.

[3] Shan – He Wu, Yu-Ming Chu: Geometric interpretation of Blundon's inequality and Ciamberlini's inequality, Journal of inequalities and Applications, December 2014

Bellow we illustrate the proof:

O246. Let P be a point inside or on the boundry of a convex polygon $A_1,A_2,\ldots,A_n$. Prove that the maximum value of $f(P)=\sum_{i=1}^{n}|P-A_i|$ is achieved when P is a vertex $A_1,A_2,\ldots,A_n$

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by G.R A. 20 Problems Solving Group, Roma, Italy

Let C be the set of points inside or on the boundry of the convexe polygon. We first note that the map $P\to f(P)$ is convex: if $\alpha,\beta\ge 0$ with $\alpha+\beta=1$ then, for any $P,Q\in C$, we have that $\alpha P+\beta Q\in C$ and

$$f(\alpha P+\beta Q)=\sum_{i=1}^{n}[\alpha(P-Ai)+\beta(Q-Ai)]$$

$$\le\alpha\sum_{i}^{n}|P-A_i|+\beta\sum_{i=1}^{n}|P-A_i|=\alpha f(P)+\beta f(Q)$$

Moreover, for any point P in C is a convex combination of the vertices $A_1,A_2,\ldots,A_n$ that is, there exists $\alpha_1,\alpha_2,\ldots,\alpha_n\ge 0$ such that $\sum_{i=1}^{n}a_i=1$ and

$$P=\sum_{i=1}^{n}\alpha_iA_i$$

Hence, by the covexity of f,

$$f(P) = f\left(\sum_{t=1}^{n} \alpha_i A_i\right) \leq \sum_{t=1}^{n} a_i\, f(A_i) \leq \sum_{t=1}^{n} a_i \max_{t=1,\dots,n} f(A_i) = \max_{t=1,\dots,n} f(A_i)$$

which means that the maximum value of $f(P)$ is attained when P is a vertex of C.