Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
This article is published with open access

Representing the Ramanujan $\frac{4}{\pi}$ Series as an Integral of Complete Elliptic Integral Functions over a Unit Length Interval

Abdulsalam Abdulhafeez Ayinde

Department of Mathematics, University of Ibadan, Ibadan

aabdulsalam030@stu.ui.edu.ng, hafeez147258369@gmail.com

October 14, 2021

Abstract

In this paper, the Ramanujan $\frac{4}{\pi}$ series is represented as an integral of the complete elliptic integral K and E functions over the interval $(0,1)$. After deriving the integrals, an integral representation for π was given in terms of the elliptic integral functions.

Mathematics Subject Classification (2020): 33Cxx, 33C75.

1 Introduction

Towards the end of the first paper that Ramanujan published in England, he wrote at the beginning of section 13, "I shall conclude this paper by giving a few series for $\frac{1}{\pi}$". Ramanujan then recorded three series representations for $\frac{1}{\pi}$.

Ramanujan proposed that [2]

$$\frac{4}{\pi}=\sum_{n=0}^{\infty}\frac{(6n+1)A_n}{4^n},\quad \text{where } A_n=\frac{\left(\frac{1}{2}\right)_n^3}{(n!)^3}$$

2 The integral representation

Let $\Omega=\sum_{n=0}^{\infty}\frac{(6n+1)\left(\frac{1}{2}\right)_n^3}{4^n (n!)^3}$

$$\implies \Omega=\underbrace{\frac{6}{\Gamma\left(\frac{1}{2}\right)^3}\sum_{n=0}^{\infty}\frac{n\Gamma\left(n+\frac{1}{2}\right)^3}{(n!)^3 4^n}}_{A}+\underbrace{\frac{1}{\Gamma\left(\frac{1}{2}\right)^3}\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)^3}{(n!)^3 4^n}}_{B}$$

$$A = \frac{6}{\Gamma\left(\frac{1}{2}\right)^6} \sum_{n=0}^{\infty} \frac{n\Gamma\left(n+\frac{1}{2}\right)^3 \Gamma\left(\frac{1}{2}\right)^3}{(n!)^3 4^n}$$
$$= \frac{6}{\pi^3} \sum_{n=0}^{\infty} \frac{n\beta\left(\left(n+\frac{1}{2}\right), \frac{1}{2}\right)^3}{4^n (n!)^3},$$

The beta function is defined as

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \text{ for } m,n > 0 \tag{1}$$

$$\implies A = \frac{6}{\pi^3} \sum_{n=0}^{\infty} \left(\frac{n}{4^n}\right) \int_0^1 x^{n-\frac{1}{2}} (1-x)^{-\frac{1}{2}} \mathrm{d}x \int_0^1 y^{n-\frac{1}{2}} (1-y)^{-\frac{1}{2}} \mathrm{d}y \int_0^1 z^{n-\frac{1}{2}} (1-z)^{-\frac{1}{2}} \mathrm{d}z$$
$$= \frac{6}{\pi^3} \int_0^1 \int_0^1 \int_0^1 \frac{((1-x)(1-y)(1-z))^{\frac{-1}{2}}}{(xyz)^{\frac{1}{2}}} \sum_{n=0}^{\infty} n \left(\frac{xyz}{4}\right)^n \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$= \frac{6}{\pi^3} \int_0^1 \int_0^1 \int_0^1 \frac{4xyz}{(4-xyz)^2} \cdot \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{((1-x)(1-y)(1-z)(xyz))^{\frac{1}{2}}}$$
$$= \frac{24}{\pi^3} \int_0^1 y^{\frac{1}{2}} (1-y)^{\frac{-1}{2}} \mathrm{d}y \int_0^1 z^{\frac{1}{2}} (1-z)^{\frac{-1}{2}} \mathrm{d}z \int_0^1 \frac{x^{\frac{1}{2}} (1-x)^{\frac{-1}{2}}}{(4-xyz)^2} \mathrm{d}x$$

Evaluating the integral inside out, let $\phi(a) = \int_0^1 \frac{x^{\frac{1}{2}} (1-x)^{\frac{-1}{2}}}{(4-ax)^2} \mathrm{d}x$ for $a = yz$.

Substitute $x = \sin^2\theta$

$$\phi(a) = \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\cos\theta} \cdot \frac{2\sin\theta.\cos\theta}{\left(4-a\sin^2\theta\right)^2} d\theta$$
$$= 2\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{\left(4-a\sin^2\theta\right)^2} d\theta$$

By Weierstrass' substitution, let $t = \tan x \implies \mathrm{d}\theta = \frac{\mathrm{d}t}{1+t^2}$, $\sin t = \frac{t}{\sqrt{1+t^2}}$

$$\implies \phi(a) = 2\int_0^{\infty} \frac{t^2}{(1+t^2)\left(4-\frac{at^2}{1+t^2}\right)^2} \cdot \frac{\mathrm{d}t}{1+t^2}$$
$$= 2\int_0^{\infty} \frac{t^2}{(4+t^2(4-a))^2} \mathrm{d}t$$
$$\overset{m\to 4-a}{=} 2\int_0^{\infty} \frac{t^2}{(4+mt^2)^2} \mathrm{d}t$$

$$\phi(4-m) = \frac{2}{m}\int_0^{\infty} \frac{\mathrm{d}t}{4+mt^2} - \frac{8}{m}\int_0^{\infty} \frac{\mathrm{d}t}{(4+mt^2)^2}$$
$$= \frac{2}{m^2}\int_0^{\infty} \frac{\mathrm{d}t}{\frac{4}{m}+t^2} - \frac{8}{m^3}\int_0^{\infty} \frac{\mathrm{d}t}{\left(\frac{4}{m}+t^2\right)^2}$$

$$
\begin{aligned}
&= \frac{2}{m^2}\frac{\sqrt{m}}{2}\arctan\left(\frac{\sqrt{m}}{2}\right)\Bigg|_0^\infty - \frac{16}{m^3\sqrt{m}}\cdot\frac{m^2}{16}\int_0^\infty \frac{\mathrm{d}t}{(t^2+1)^2}\\
&\overset{t=\tan\alpha}{=} \frac{\pi}{2m\sqrt{m}} - \frac{1}{m\sqrt{m}}\int_0^{\frac{\pi}{2}} \frac{\sec^2\alpha}{\sec^4\alpha}\mathrm{d}\alpha\\
&= \frac{\pi}{2m\sqrt{m}} - \frac{1}{m\sqrt{m}}\int_0^{\frac{\pi}{2}} \cos^2\alpha\,\mathrm{d}\alpha\\
&= \frac{\pi}{2m\sqrt{m}} - \frac{1}{m\sqrt{m}}\left(\frac{\alpha}{2}+\frac{\sin 2\alpha}{4}\right)\Bigg|_0^{\frac{\pi}{2}}\\
&= \frac{\pi}{2m\sqrt{m}} - \frac{\pi}{4m\sqrt{m}} = \frac{\pi}{4m\sqrt{m}}
\end{aligned}
$$

$$
\implies \phi(a) = \frac{\pi}{4(4-a)\sqrt{4-a}}
$$

$$
A = \frac{24}{\pi^3}\int_0^1 y^{\frac{1}{2}}(1-y)^{\frac{-1}{2}}\,\mathrm{d}y\int_0^1 z^{\frac{1}{2}}(1-z)^{\frac{-1}{2}}\cdot\frac{\pi}{4(4-yz)\sqrt{4-yz}}\mathrm{d}z
$$

By Fubini's theorem, we have that,

$$
A = \frac{6}{\pi^2}\int_0^1 z^{\frac{1}{2}}(1-z)^{\frac{-1}{2}}\,\mathrm{d}z\int_0^1 y^{\frac{1}{2}}(1-y)^{\frac{-1}{2}}\cdot\frac{\mathrm{d}y}{(4-yz)\sqrt{4-yz}}
$$

Let $\Phi(z) = \displaystyle\int_0^1 \frac{y^{\frac{1}{2}}(1-y)^{\frac{-1}{2}}}{(4-yz)^{\frac{3}{2}}}\mathrm{d}y$

Substitute $y = \sin^2\theta$

$$
\begin{aligned}
\Phi(z) &= \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\cos\theta}\cdot\frac{2\sin\theta\cdot\cos\theta}{\left(4-z\sin^2\theta\right)^{\frac{3}{2}}}\mathrm{d}\theta\\
&= 2\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{\left(4-z\sin^2\theta\right)^{\frac{3}{2}}}\mathrm{d}\theta\\
&\overset{z\to 4z}{=} \frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{\left(1-z\sin^2\theta\right)^{\frac{3}{2}}}\mathrm{d}\theta\\
&= \frac{1}{4z}\left(\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\left(1-z\sin^2\theta\right)^{\frac{3}{2}}} - \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\left(1-z\sin^2\theta\right)^{\frac{1}{2}}}\right)
\end{aligned}
$$

It can be proven by Leibnitz-Maclaurin series that

$$
\frac{1}{(1-x)^a} = \sum_{k=0}^{\infty}\frac{(a)_k x^k}{k!}
$$

$$
\Phi(z) = \frac{1}{4z}\int_0^{\frac{\pi}{2}}\sum_{k=0}^{\infty}\frac{z^k\sin^{2k}\theta\left(\frac{3}{2}\right)_k}{k!}\mathrm{d}\theta - \frac{1}{4z}\int_0^{\frac{\pi}{2}}\sum_{k=0}^{\infty}\frac{z^k\sin^{2k}\theta\left(\frac{1}{2}\right)_k}{k!}\mathrm{d}\theta
$$

Interchanging the summation for integration and integrating term-wise

$$
\Phi(z) = \frac{1}{4z}\sum_{k=0}^{\infty}\frac{z^k\left(\frac{3}{2}\right)_k}{k!}\int_0^{\frac{\pi}{2}}\sin^{2k}\theta\mathrm{d}\theta - \frac{1}{4z}\sum_{k=0}^{\infty}\frac{z^k\left(\frac{1}{2}\right)_k}{k!}\int_0^{\frac{\pi}{2}}\sin^{2k}\theta\mathrm{d}\theta
$$

By substituting $x = \sin^2\theta$ in (1),

$$\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, \mathrm{d}\theta$$

$$2\int_0^{\frac{\pi}{2}} \sin^{2k}\theta \, \mathrm{d}\theta = \beta\left(k+\frac{1}{2},\frac{1}{2}\right)$$

$$\beta\left(k+\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)^2}{\Gamma\left(\frac{1}{2}\right)\Gamma(k+1)} = \frac{\left(\frac{1}{2}\right)_k \cdot \pi}{(1)_k}$$

$$\implies \Phi(z) = \frac{\pi}{8z}\sum_{k=0}^{\infty} \frac{\left(\frac{3}{2}\right)_k\left(\frac{1}{2}\right)_k}{(1)_k}\cdot\frac{z^k}{k!} - \frac{\pi}{8z}\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_k}{(1)_k}\cdot\frac{z^k}{k!} \tag{2}$$

Writing the series in (2) in terms of Gauss' Hypergeometric function,

$$\Phi(z) = \frac{\pi}{8z}\,{}_2F_1\left(\frac{3}{2},\frac{1}{2};1;z\right) - \frac{\pi}{8z}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;z\right)$$

The Elliptic Integral functions are defined as;

$$F(k,\theta) = \int_0^{\theta} \frac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}}, \;\; E(k,\theta) = \int_0^{\theta} \sqrt{1-k^2\sin^2\vartheta}\, d\vartheta, \text{ for } 0<k<1 \wedge 0<\theta\le\frac{\pi}{2}.$$

When $\theta = \frac{\pi}{2}$, the integral is said to be complete. At $\theta = \frac{\pi}{2}$, $F(k,\theta)$ is denoted as $K(k)$

$$\begin{aligned} K(k) &= \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \overset{t\to\sin\theta}{=} \int_0^1 \frac{\mathrm{d}t}{\sqrt{1-k^2t^2}\sqrt{1-t^2}} \\ &\overset{t\to u^2}{=} \frac{1}{2}\int_0^1 \frac{u^{-\frac{1}{2}}}{\sqrt{1-uk^2}\sqrt{1-u}}\mathrm{d}u \end{aligned} \tag{3}$$

Gauss' Hypergeometric function Integral representation is [1]

$$\beta(c-b,b)\,{}_2F_1(a,b;c;z) = \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a}\,\mathrm{d}t \tag{4}$$

By comparing (4) with (3),

$$b-1 = -\frac{1}{2},\; c-b-1 = -\frac{1}{2},\; z = k^2 \implies a = b = \frac{1}{2}, c = 1\; z = k^2.$$

$$\begin{aligned} K(k) &= \frac{1}{2}\beta\left(1-\frac{1}{2},\frac{1}{2}\right)\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right) \\ &= \frac{1}{2}\beta\left(\frac{1}{2},\frac{1}{2}\right)\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right) \end{aligned}$$

$$\left\{\beta\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \pi\right\}$$

$$= \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right)$$

Similarly,

$$E(k) = \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},-\frac{1}{2};1;k^2\right)$$

By Euler's transformation;

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c;z)$$

$$\implies {}_2F_1\left(\frac{3}{2},\frac{1}{2};1;k^2\right) = \frac{1}{1-z}\,{}_2F_1\left(\frac{1}{2},-\frac{1}{2};1;k^2\right)$$

$$\begin{aligned}
\Phi(z) &= \frac{\pi}{8z}\,{}_2F_1\left(\frac{3}{2},\frac{1}{2};1;z\right) - \frac{\pi}{8z}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;z\right)\\
&= \frac{E(\sqrt{z})}{4z(1-z)} - \frac{K(\sqrt{z})}{4z}\\
&\overset{z\to\frac{z}{4}}{=} \frac{4E\left(\sqrt{\frac{z}{4}}\right)}{z(4-z)} - \frac{K\left(\sqrt{\frac{z}{4}}\right)}{z}
\end{aligned}$$

$$\begin{aligned}
\implies A &= \frac{6}{\pi^2}\int_0^1 z^{\frac{1}{2}}(1-z)^{\frac{-1}{2}}\cdot\Phi(z)\,\mathrm{d}z\\
&= \frac{6}{\pi^2}\int_0^1 z^{\frac{1}{2}}(1-z)^{\frac{-1}{2}}\left(\frac{4E\left(\sqrt{\frac{z}{4}}\right)}{z(4-z)} - \frac{K\left(\sqrt{\frac{z}{4}}\right)}{z}\right)\mathrm{d}z\\
&= \frac{6}{\pi^2}\int_0^1 \frac{4E\left(\sqrt{\frac{z}{4}}\right) - 4K\left(\sqrt{\frac{z}{4}}\right) + zK\left(\sqrt{\frac{z}{4}}\right)}{(4-z)\sqrt{z-z^2}}\,\mathrm{d}z
\end{aligned}$$

$$\begin{aligned}
B &= \frac{1}{\Gamma\left(\frac{1}{2}\right)^3}\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)^3}{(n!)^3\,4^n}\\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)^6}\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)^3\Gamma\left(\frac{1}{2}\right)^3}{(n!)^3\,4^n}\\
&\overset{\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}}{=} \frac{1}{\pi^3}\sum_{n=0}^{\infty}\frac{\beta\left(\left(n+\frac{1}{2}\right),\frac{1}{2}\right)^3}{4^n\,(n!)^3}\\
&= \frac{1}{\pi^3}\sum_{n=0}^{\infty}\left(\frac{1}{4^n}\right)\int_0^1 x^{n-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\,\mathrm{d}x\int_0^1 y^{n-\frac{1}{2}}(1-y)^{-\frac{1}{2}}\,\mathrm{d}y\int_0^1 z^{n-\frac{1}{2}}(1-z)^{-\frac{1}{2}}\,\mathrm{d}z\\
&= \frac{1}{\pi^3}\int_0^1\int_0^1\int_0^1 \frac{((1-x)(1-y)(1-z))^{\frac{-1}{2}}}{(xyz)^{\frac{1}{2}}}\sum_{n=0}^{\infty}\left(\frac{xyz}{4}\right)^n\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\
&= \frac{1}{\pi^3}\int_0^1\int_0^1\int_0^1 \frac{4}{4-xyz}\cdot\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{((1-x)(1-y)(1-z)(xyz))^{\frac{1}{2}}}\\
&= \frac{4}{\pi^3}\int_0^1 y^{\frac{-1}{2}}(1-y)^{\frac{-1}{2}}\,\mathrm{d}y\int_0^1 z^{\frac{-1}{2}}(1-z)^{\frac{-1}{2}}\,\mathrm{d}z\int_0^1\frac{x^{\frac{-1}{2}}(1-x)^{\frac{-1}{2}}}{4-xyz}\,\mathrm{d}x
\end{aligned}$$

Evaluating the integral inside out, let $\varphi(a) = \int_0^1 \frac{x^{\frac{-1}{2}}(1-x)^{\frac{-1}{2}}}{4-ax}\mathrm{d}x$ for $a = yz$.
Substitute $x = \sin^2\theta$

$$\begin{aligned}\varphi(a) &= \int_0^{\frac{\pi}{2}} \frac{1}{\sin\theta\cdot\cos\theta}\cdot\frac{2\sin\theta.\cos\theta}{4-a\sin^2\theta}\mathrm{d}\theta\\ &= 2\int_0^{\frac{\pi}{2}} \frac{1}{4-a\sin^2\theta}\mathrm{d}\theta\end{aligned}$$

By Weierstrass' substitution, let $t = \tan x \implies \mathrm{d}\theta = \frac{\mathrm{d}t}{1+t^2}, \sin t = \frac{t}{\sqrt{1+t^2}}$.

$$\begin{aligned}\varphi(a) &= 2\int_0^{\infty} \frac{1}{\left(4-\frac{at^2}{1+t^2}\right)}\cdot\frac{dt}{1+t^2}\\ &= 2\int_0^{\infty}\frac{dt}{(4+t^2(4-a))}\\ &\overset{m\to 4-a}{=} 2\int_0^{\infty}\frac{dt}{(4+mt^2)}\\ \varphi(4-m) &= \frac{2}{m}\int_0^{\infty}\frac{dt}{\frac{4}{m}+t^2}\\ &= \frac{2}{m}\cdot\frac{\sqrt{m}}{2}\arctan\left(\frac{\sqrt{m}}{2}\right)\Big|_0^{\infty}\\ &= \frac{2}{m}\cdot\frac{\sqrt{m}}{2}\cdot\frac{\pi}{2} = \frac{\pi}{2\sqrt{m}}\\ \varphi(a) &= \frac{\pi}{2\sqrt{4-a}}\end{aligned}$$

$$\begin{aligned}\implies B &= \frac{4}{\pi^3}\cdot\frac{\pi}{2}\int_0^1 y^{\frac{-1}{2}}(1-y)^{\frac{-1}{2}}\,\mathrm{d}x\\ &= \int_0^1 z^{\frac{-1}{2}}(1-z)^{\frac{-1}{2}}\cdot\frac{\mathrm{d}z}{\sqrt{4-yz}}\\ &= \frac{2}{\pi^2}\int_0^1 z^{\frac{-1}{2}}(1-z)^{\frac{-1}{2}}\,\mathrm{d}z\int_0^1 y^{\frac{-1}{2}}(1-y)^{\frac{-1}{2}}\cdot\frac{\mathrm{d}y}{\sqrt{4-yz}}\end{aligned}$$

Let $\chi(z) = \int_0^1 \frac{y^{\frac{-1}{2}}(1-y)^{\frac{-1}{2}}}{(4-yz)^{\frac{1}{2}}}\mathrm{d}y$
Substitute $y = \sin^2\theta$

$$\begin{aligned}\chi(z) &= \int_0^{\frac{\pi}{2}}\frac{1}{\sin\theta\cos\theta}\cdot\frac{2\sin\theta\cdot\cos\theta}{\left(4-z\sin^2\theta\right)^{\frac{1}{2}}}\mathrm{d}\theta\\ &= 2\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{4-z\sin^2\theta}} \overset{z\to 4z}{=} \frac{1}{2}\cdot 2\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-z\sin^2\theta}}\\ &= K\left(\sqrt{z}\right) \overset{z\to\frac{z}{4}}{=} K\left(\sqrt{\frac{z}{4}}\right)\end{aligned}$$

$$\implies B = \frac{2}{\pi^2}\int_0^1 z^{\frac{-1}{2}}(1-z)^{\frac{-1}{2}} K\left(\sqrt{\frac{z}{4}}\right)\mathrm{d}z = \frac{2}{\pi^2}\int_0^1 \frac{K\left(\sqrt{\frac{z}{4}}\right)}{\sqrt{z-z^2}}\mathrm{d}z$$

Recall that $\Omega = A + B$

$$\begin{aligned}\implies \Omega &= \frac{6}{\pi^2}\int_0^1 \frac{4E\left(\sqrt{\frac{z}{4}}\right)-4K\left(\sqrt{\frac{z}{4}}\right)+zK\left(\sqrt{\frac{z}{4}}\right)}{(4-z)\sqrt{z-z^2}}\mathrm{d}z + \frac{2}{\pi^2}\int_0^1 \frac{K\left(\sqrt{\frac{z}{4}}\right)}{\sqrt{z-z^2}}\mathrm{d}z \\ &= \int_0^1 \frac{24E\left(\sqrt{\frac{z}{4}}\right)+4(z-4)K\left(\sqrt{\frac{z}{4}}\right)}{\pi^2(4-z)\sqrt{z-z^2}}\mathrm{d}z\end{aligned}$$

$$\therefore \sum_{n=0}^{\infty}\frac{(6n+1)\left(\frac{1}{2}\right)_n^3}{4^n(n!)^3} = \frac{4}{\pi^2} = \int_0^1 \frac{24E\left(\sqrt{\frac{z}{4}}\right)+4(z-4)K\left(\sqrt{\frac{z}{4}}\right)}{\pi^2(4-z)\sqrt{z-z^2}}\mathrm{d}z$$

$$\begin{aligned}\implies \frac{4}{\pi} &= \int_0^1 \frac{24E\left(\sqrt{\frac{z}{4}}\right)+4(z-4)K\left(\sqrt{\frac{z}{4}}\right)}{\pi^2(4-z)\sqrt{z-z^2}}\mathrm{d}z \\ \implies 4\pi &= \int_0^1 \frac{24E\left(\sqrt{\frac{z}{4}}\right)+4(z-4)K\left(\sqrt{\frac{z}{4}}\right)}{(4-z)\sqrt{z-z^2}}\mathrm{d}z \\ \implies \pi &= \int_0^1 \frac{6E\left(\sqrt{\frac{z}{4}}\right)+(z-4)K\left(\sqrt{\frac{z}{4}}\right)}{(4-z)\sqrt{z-z^2}}\mathrm{d}z\end{aligned}$$

$E\left(\sqrt{z}\right)$ and $K\left(\sqrt{z}\right)$ are complete elliptic integrals with z as the elliptic modulus and $(a)_n$, the Pochhammer symbol is defined as the rising factorial i.e. $(a)_n = \dfrac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)(a+2)(a+3)(a+4)\cdots(a+n-1)$.

3 Main Results

$$\frac{4}{\pi} = \int_0^1 \frac{24E\left(\sqrt{\frac{z}{4}}\right)+4(z-4)K\left(\sqrt{\frac{z}{4}}\right)}{\pi^2(4-z)\sqrt{z-z^2}}\mathrm{d}z$$

$$\pi = \int_0^1 \frac{6E\left(\sqrt{\frac{z}{4}}\right)+(z-4)K\left(\sqrt{\frac{z}{4}}\right)}{(4-z)\sqrt{z-z^2}}\mathrm{d}z$$

References

[1] Zaid Alyafeai. *Advanced Inegral Techniques*.

[2] BRUCE C.BERNDT, S.BHARGAVA, and FRANK G. GARVAN. *RAMANUJAN'S THEORIES OF ELLIPTIC FUNCTIONS TO ALTERNATIVE BASES*.