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If $f:\mathbb{R}\to(0,\infty)$, $a,b\in\mathbb{R}$, $a\le b$ then:

$$\int_a^b\int_a^b\left((f(x)+f(y))\tan^{-1}\left(\frac{f(x)+f(y)}{2}\right)\right)dx\,dy\le 2(b-a)\int_a^b\left(f(x)\tan^{-1}(f(x))\right)dx$$

Proposed by Daniel Sitaru – Romania

Solution by Florentin Vișescu-Romania

$$f(t)=\arctan t\,;f'(t)=\arctan t+\frac{t}{1+t^2}$$

$$f''(t)=\frac{1}{1+t^2}+\frac{1+t^2-2t^2}{(1+t^2)^2}=\frac{1}{1+t^2}+\frac{1-t^2}{(1+t^2)^2}=\frac{1+t^2-t^2}{(1+t^2)^2}$$

$$=\frac{2}{(1+t^2)^2}>0\Rightarrow f\text{ **convexe**}\Rightarrow f\left(\frac{x+y}{2}\right)\le\frac{f(x)+f(y)}{2}$$

$$\frac{x+y}{2}\arctan\frac{x+y}{2}\le\frac{1}{2}(x\arctan x+y\arctan y)$$

$$(x+y)\arctan\frac{x+y}{2}\le x\arctan x+y\arctan y\,;\forall x,y\in\mathbb{R}$$

$$x\to f(x);\,y\to f(y)$$

$$\int_a^b\int_a^b[f(x)+f(y)]\arctan\frac{f(x)+f(y)}{2}dx\,dy\le\int_a^b\int_a^b f(x)\arctan(x)+f(y)\arctan(y)\,dx\,dy$$

$$\int_a^b \int_a^b \left(f(x)+f(y)\right)\arctan\frac{f(x)+f(y)}{2}\,dx\,dy \le 2(b-a)\int_a^b f(x)\arctan(x)\,dx$$