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Let f be continuous on $[0,1]$. If $af(b)+bf(a)\leq 2;\forall b\in[0,1]$, prove:

$$\int_0^1 f(x)\,dx\leq\frac{\pi}{2}$$

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Suppose: $r=\sqrt{a^2+b^2}$. *So:*

$$\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2+\left(\frac{b}{\sqrt{a^2+b^2}}\right)^2=\frac{a^2+b^2}{a^2+b^2}=1$$

So: $\exists\theta\in\mathbb{R}:\cos\theta=\frac{a}{\sqrt{a^2+b^2}},\sin\theta=\frac{b}{\sqrt{a^2+b^2}}$

$$\cos\theta f(b)+\sin\theta f(a)\leq\frac{2}{\sqrt{a^2+b^2}}$$

For: $b=\sin\theta,a=\cos\theta\in[0,1]$. *Then:*

$$\cos\theta f(\sin\theta)+\sin\theta f(\cos\theta)\leq 2$$

$$I_1=\int_0^1 f(x)\,dx=\int_0^{\frac{\pi}{2}} f(\sin\theta)\cdot\cos\theta\,d\theta\begin{cases}x=\sin\theta\\dx=\cos\theta\,d\theta\end{cases}\begin{cases}x=0\Rightarrow\theta=0\\x=1\Rightarrow\theta=\frac{\pi}{2}\end{cases}$$

$$I_2 = \int_0^1 f(x)\,dx = \int_{\frac{\pi}{2}}^{0} -f(\cos\theta)\sin\theta\,d\theta \begin{cases} y = \cos\theta \\ dy = \sin\theta\,d\theta \end{cases}$$

$$\begin{cases} y = 0 \Rightarrow \theta = \dfrac{\pi}{2} \\ y = 1 \Rightarrow \theta = 0 \end{cases}$$

$$2\int_0^1 f(x)\,dx = \int_0^{\frac{\pi}{2}} f(\sin\theta)\cos\theta\,d\theta + \int_0^{\frac{\pi}{2}} f(\cos\theta)\sin\theta\,d\theta \le 2\int_0^{\frac{\pi}{2}} d\theta = 2[\theta]_0^{\frac{\pi}{2}} = \pi$$

$$\int_0^1 f(x)\,dx \le \frac{\pi}{2}$$