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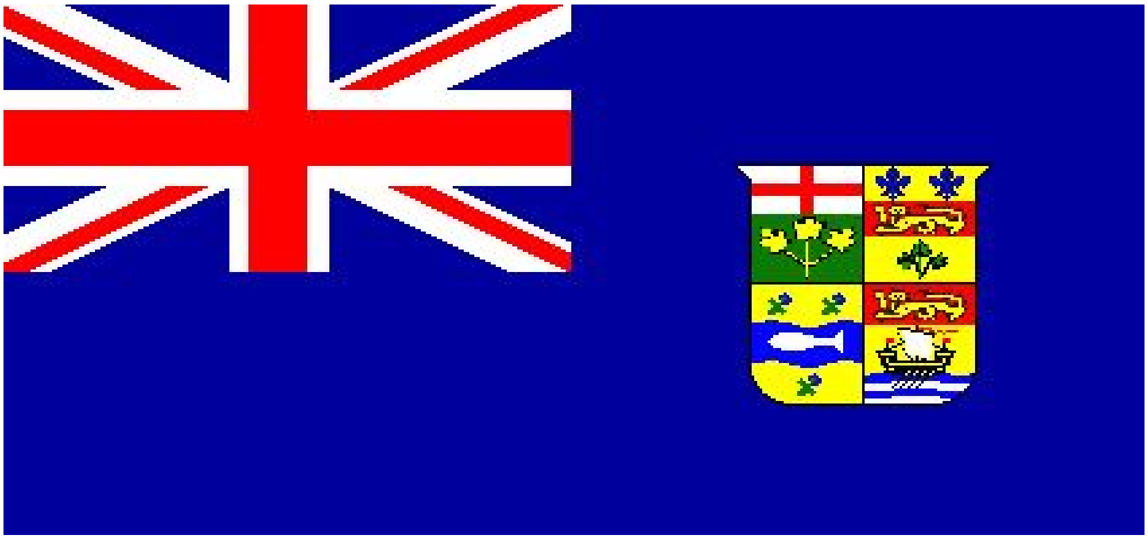

Prove without softs:

$$\int_0^1\int_0^1\int_0^1 \frac{z\sqrt{xy}}{(x^2+1)(y+1)}\,dxdydz < \frac{1}{6}$$

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Solution by Soumava Chakraborty-Kolkata-India

Proof : Let $\sqrt{x} = \theta \therefore x = \theta^2 \Rightarrow dx = 2\theta d\theta$ and let $\sqrt{y} = \varphi \therefore y = \varphi^2 \Rightarrow dy = 2\varphi d\varphi$

$$\therefore \int_0^1\int_0^1\int_0^1 \frac{z\sqrt{xy}}{(x^2+1)(y+1)}\,dxdydz \overset{(1)}{=} 4\int_0^1\int_0^1\int_0^1 \frac{z\theta^2\varphi^2}{(\theta^4+1)(\varphi^2+1)}\,d\theta d\varphi dz$$

Let $\Omega = \int_0^1 \frac{\theta^2 d\theta}{\theta^4+1}$ Let $\theta^2 = \tan\omega \left(0 \le \omega \le \frac{\pi}{4}\right) \therefore 2\theta d\theta = \sec^2\omega d\omega \Rightarrow d\theta = \frac{\sec^2\omega d\omega}{2\sqrt{\tan\omega}}$

$$\therefore \Omega = \int_0^{\frac{\pi}{4}} \frac{\tan\omega\sec^2\omega d\omega}{2\sec^2\omega\sqrt{\tan\omega}} = \frac{1}{2}\int_0^{\frac{\pi}{4}} \sqrt{\tan\omega}\,d\omega \Rightarrow \Omega \overset{(2)}{=} \frac{1}{2}\int_0^{\frac{\pi}{4}} \sqrt{\tan\omega}\,d\omega$$

$$\text{Let } I = \int \sqrt{\tan\omega}\,d\omega \text{ and let } J = \int \sqrt{\cot\omega}\,d\omega$$

$$I + J = \int (\sqrt{\tan\omega} + \sqrt{\cot\omega})d\omega = \sqrt{2}\int \frac{(\sin\omega + \cos\omega)}{\sqrt{2\sin\omega\cos\omega}}d\omega$$
$$= -\sqrt{2}\int \frac{-(\sin\omega + \cos\omega)d\omega}{\sqrt{1 - (\cos\omega - \sin\omega)^2}} = -\sqrt{2}\int \frac{dt}{\sqrt{1 - t^2}}(t = \cos\omega - \sin\omega)$$
$$\overset{(a)}{\cong} -\sqrt{2}\sin^{-1}(\cos\omega - \sin\omega) + C$$
$$I - J = \int (\sqrt{\tan\omega} - \sqrt{\cot\omega})d\omega = \sqrt{2}\int \frac{(\sin\omega - \cos\omega)}{\sqrt{2\sin\omega\cos\omega}}d\omega$$
$$= -\sqrt{2}\int \frac{(\cos\omega - \sin\omega)d\omega}{\sqrt{(\sin\omega + \cos\omega)^2 - 1}} = -\sqrt{2}\int \frac{dm}{\sqrt{m^2 - 1}}(t = \sin\omega + \cos\omega)$$
$$\overset{(b)}{\cong} -\sqrt{2}\ln\left|\sin\omega + \cos\omega + \sqrt{2\sin\omega\cos\omega}\right| + K$$
$$(a) + (b) \Rightarrow 2I = 2\int \sqrt{\tan\omega}\,d\omega$$
$$= -\sqrt{2}\sin^{-1}(\cos\omega - \sin\omega) - \sqrt{2}\ln\left|\sin\omega + \cos\omega + \sqrt{2\sin\omega\cos\omega}\right| + C + K$$
$$\therefore 2\int_0^{\frac{\pi}{4}} \sqrt{\tan\omega}\,d\omega$$
$$= -\sqrt{2}\sin^{-1}\left(\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right) - \sqrt{2}\ln\left|\sin\frac{\pi}{4} + \cos\frac{\pi}{4} + \sqrt{2\sin\frac{\pi}{4}\cos\frac{\pi}{4}}\right| + \sqrt{2}\sin^{-1}(\cos 0 - \sin 0) + \sqrt{2}\ln\left|\sin 0 + \cos 0 + \sqrt{2\sin 0\cos 0}\right|$$
$$= -\sqrt{2}\ln(\sqrt{2} + 1) + \sqrt{2}\left(\frac{\pi}{2}\right) \Rightarrow \frac{1}{2}\int_0^{\frac{\pi}{4}} \sqrt{\tan\omega}\,d\omega \overset{(3)}{\cong} \frac{\sqrt{2}}{4}\left(\frac{\pi}{2} - \ln(\sqrt{2} + 1)\right) \text{ So, (2), (3)}$$
$$\Rightarrow \int_0^1 \frac{\theta^2 d\theta}{\theta^4 + 1} \overset{(i)}{\cong} \frac{\sqrt{2}}{4}\left(\frac{\pi}{2} - \ln(\sqrt{2} + 1)\right)$$
$$\text{Again, } \int_0^1 \frac{\varphi^2 d\varphi}{\varphi^2 + 1} = \int_0^1 \frac{(\varphi^2 + 1)d\varphi}{\varphi^2 + 1} - \int_0^1 \frac{d\varphi}{\varphi^2 + 1} = 1 - \tan^{-1}1 = \frac{4 - \pi}{4}$$
$$\Rightarrow \int_0^1 \frac{\varphi^2 d\varphi}{\varphi^2 + 1} \overset{(ii)}{\cong} \frac{4 - \pi}{4}$$

$$(1),(i) \Rightarrow \int_0^1\int_0^1\int_0^1 \frac{z\sqrt{xy}}{(x^2+1)(y+1)}dxdydz = 4\left(\frac{\sqrt{2}}{4}\right)\left(\frac{\pi}{2}-\ln(\sqrt{2}+1)\right)\int_0^1\int_0^1 \frac{z\varphi^2 d\varphi dz}{\varphi^2+1}$$

$$= \sqrt{2}\left(\frac{\pi}{2}-\ln(\sqrt{2}+1)\right)\left(\frac{4-\pi}{4}\right)\int_0^1 zdz$$

$$= \boxed{\frac{\sqrt{2}}{8}\left(\frac{\pi}{2}-\ln(\sqrt{2}+1)\right)(4-\pi)} \approx .1046 < \frac{1}{6}\,(\text{Proved})$$