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$$\Omega_1(t) = \int_0^1 \left(\frac{x^{\sin^2 t}}{1 + x^{\cos^2 t}} \right) dt, \Omega_2(t) = \int_0^1 \left(\frac{x^{\cos^2 t}}{1 + x^{\sin^2 t}} \right) dt$$

Prove that:

$$\Omega_1(t) + \Omega_2(t) \geq \frac{4 - \sin 2t}{8 + \sin 2t}, t \in \left[0, \frac{\pi}{2}\right]$$

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Solution by Şerban George Florin – Romania

$$\begin{aligned} \frac{x^{\sin^2 t}}{1 + x^{\cos^2 t}} + \frac{x^{\cos^2 t}}{1 + x^{\sin^2 t}} &\geq 2 \sqrt{\frac{x^{\sin^2 t} + \cos^2 t}{(1 + x^{\cos^2 t})(1 + x^{\sin^2 t})}} = \\ &= 2 \sqrt{\frac{x}{(1 + x^{\cos^2 t})(1 + x^{\sin^2 t})}} \geq 2 \sqrt{\frac{x}{(1 + 1)(1 + 1)}} = \frac{2\sqrt{x}}{2} = \sqrt{x} \end{aligned}$$

because: $x^{\cos^2 t} \leq 1 = x^0, (\forall)x \in [0, 1], 0 \leq \cos^2 t \leq 1$ and

$$x^{\sin^2 t} \leq 1 = x^0, (\forall)x \in [0, 1], 0 \leq \sin^2 t \leq 1$$

$$\begin{aligned} \Rightarrow \Omega_1(t) + \Omega_2(t) &= \int_0^1 \left(\frac{x^{\sin^2 t}}{1 + x^{\cos^2 t}} + \frac{x^{\cos^2 t}}{1 + x^{\sin^2 t}} \right) dt \geq \int_0^1 \sqrt{x} dt = \\ &= \int_0^1 t^{\frac{1}{2}} dt = \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} \Big|_0^1 = \frac{t^{\frac{3}{2}}}{\frac{3}{2}} \Big|_0^1 = \frac{2}{3} \geq \frac{4 - \sin 2t}{8 + \sin 2t} \end{aligned}$$

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$$\sin 2t = y \geq 0, t \in \left[0, \frac{\pi}{2}\right] \Rightarrow 2t \in [0, \pi] \Rightarrow 8 + y > 0 \text{ and}$$

$$4 - y > 0 \text{ because } y \in [0, 1]$$

$$\Rightarrow 2(8 + y) \geq 3 \cdot (4 - y), 16 + 2y \geq 12 - 3y$$

$$\Rightarrow 4 + 5y \geq 0, \text{ true, because } y \geq 0$$