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In acute ΔABC the following relationship holds:

$$\left(\sum_{cyc} a^m \sin^m A\right)\left(\sum_{cyc}\left(\frac{a}{s-a}\right)^q\right) \geq \frac{9 \cdot 2^{m+q} \cdot s^{m+n}}{3^{m+n} \cdot R^n}, m, n, q \geq 2$$

Proposed by Radu Diaconu-Romania

Solution by Șerban George Florin-Romania

$$a \leq b \leq c \Rightarrow \sin A \leq \sin B \leq \sin C$$

$$a^m \leq b^m \leq c^m \Rightarrow \sin^n A \leq \sin^n B \leq \sin^n C$$

$$\sum_{cyc} a^m \sin^n A \overset{Chebyshev}{\geq} \frac{1}{3}\left(\sum a^m\right)\left(\sum \sin^n A\right) \overset{Holder}{\geq} \frac{1}{3} \frac{(\sum a)^m}{3^{m-1}} \cdot \frac{(\sum \sin A)^n}{3^{n-1}}$$

$$= \frac{(2s)^m \cdot \left(\frac{a+b+c}{2R}\right)^n}{3^{m+n-1}} = \frac{2^m \cdot s^m \cdot 2^n \cdot s^n}{2^n \cdot R^n \cdot 3^{m+n-1}} = \frac{2^m \cdot s^{m+n}}{R^n \cdot 3^{m+n-1}}$$

$$\left(\sum_{cyc}\left(\frac{a}{s-a}\right)^q\right) \overset{Holder}{\geq} \frac{\left(\sum \frac{a}{s-a}\right)^q}{3^{q-1}} = \frac{\left[\frac{2(2R-r)}{r}\right]^q}{3^{q-1}} \geq \frac{6^q}{3^{q-1}} = 3 \cdot 2^q$$

Because (Euler) $\frac{2(2R-r)}{r} \geq 6 \Rightarrow 4R - 2r \geq 6r \Rightarrow 4R \geq 8r, R \geq 2r$

$$\Rightarrow\left(\sum_{cyc} a^m \sin^n A\right)\cdot\left(\sum\left(\frac{a}{s-a}\right)^q\right)\geq\frac{2^m\cdot s^{m+n}}{R^n\cdot 3^{m+n-1}}\cdot 3\cdot 2^q =$$

$$=\frac{2^{m+q}\cdot s^{m+n}\cdot 9}{R^n\cdot 3^{m+n}}$$, ***true.***