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ROMANIAN MATHEMATICAL MAGAZINE

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A TERESHIN TYPE INEQUALITY BY BOGDAN FUȘTEI

In ΔABC the following relationship holds:

$$m_a \geq \frac{n_a^2 + g_a^2 + 2rr_a}{4R}, \quad n_a - \text{Nagel's cevian}, \quad g_a - \text{Gergonne's cevian}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \text{Stewart's theorem} &\Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c) \\ &\Rightarrow s(b^2 + c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc) \\ &\Rightarrow s(b^2 + c^2) - 2sbc = an_a^2 + a(as - s^2) \Rightarrow s(b^2 + c^2 - a^2 - 2bc) = an_a^2 - as^2 \\ &\Rightarrow an_a^2 = as^2 + s(2bc \cos A - 2bc) = as^2 - 4sbc \sin^2 \frac{A}{2} \\ &= as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)} = as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a \left(\frac{2\Delta}{a} \right) \left(\frac{\Delta}{s-a} \right) \\ &= as^2 - 2ah_a r_a \Rightarrow n_a^2 \stackrel{(1)}{=} s^2 - 2h_a r_a \end{aligned}$$

$$\begin{aligned} \text{Again, Stewart's theorem} &\Rightarrow b^2(s-b) + c^2(s-c) = ag_a^2 + a(s-b)(s-c) \\ &\Rightarrow s(b^2 + c^2) - (b^3 + c^3) = ag_a^2 + a(s^2 - s(2s-a) + bc) = ag_a^2 + a(-s^2 + as + bc) \\ &\Rightarrow ag_a^2 = as^2 + \frac{(b^2 + c^2)(\sum a) - 2(b^3 + c^3) - a^2(\sum a) - 2abc}{2} \end{aligned}$$

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$$\begin{aligned}
 &= as^2 + \frac{ab^2 + ac^2 + b^3 + bc^2 + b^2c + c^3 - 2(b^3 + c^3) - a^3 - a^2b - a^2c - 2abc}{2} \\
 &= as^2 + \frac{a(b-c)^2 - (a^3 + b^3 + c^3) + b^2c + bc^2 - a^2b - a^2c}{2} \\
 &= as^2 + \frac{a(b-c)^2 - a^2(\sum a) - (b+c)(b^2 - bc + c^2) + bc(b+c)}{2} \\
 &= as^2 + \frac{a(b-c)^2 - 2sa^2 - (2s-a)(b-c)^2}{2} \\
 &= as^2 + \frac{2a(b-c)^2 - 2sa^2 - 2s(b^2 + c^2 - 2bc)}{2} = as^2 + a(b-c)^2 - s \sum a^2 + 2sbc \\
 &\Rightarrow g_a^2 \stackrel{(2)}{=} (b-c)^2 + s^2 - \frac{s \sum a^2}{a} + \frac{2sbc}{a} \\
 &\quad (1)+(2) \Rightarrow n_a^2 + g_a^2 + 2r_b r_c \\
 &= 2s^2 + (b-c)^2 - \frac{s \sum a^2}{a} + \frac{2sbc}{a} - 2 \left(\frac{2\Delta}{a} \right) \left(\frac{\Delta}{s-a} \right) + 2 \frac{\Delta^2}{(s-b)(s-c)} \\
 &= 2s^2 + (b-c)^2 - \frac{s \sum a^2}{a} + \frac{2sbc}{a} - \frac{4s(s-a)(s-b)(s-c)}{a(s-a)} + \frac{2s(s-a)(s-b)(s-c)}{(s-b)(s-c)} \\
 &= (b-c)^2 + 2s(s-a) + 2s^2 - s \left\{ \frac{\sum a^2 + 4(s-b)(s-c) - 2bc}{a} \right\} \\
 &= (b-c)^2 + 2s(s-a) + 2s^2 - s \left\{ \frac{a^2 - (b-c)^2 + a^2 + (b^2 + c^2 - 2bc)}{a} \right\} \\
 &= (b-c)^2 + 2s(s-a) + 2s^2 - s \left(\frac{2a^2}{a} \right) = (b-c)^2 + 4s(s-a) \\
 &= (b-c)^2 + (b+c+a)(b+c-a) \\
 &= (b-c)^2 + (b+c)^2 - a^2 = 2b^2 + 2c^2 - a^2 = 4m_a^2 \\
 &\therefore n_a^2 + g_a^2 + 2r_b r_c = 4m_a^2 \Rightarrow n_a^2 + g_a^2 + 2rr_a = 4m_a^2 - 2r_b r_c + 2rr_a \\
 &= 2b^2 + 2c^2 - a^2 - \frac{2s(s-a)(s-b)(s-c)}{(s-b)(s-c)} + \frac{2s(s-a)(s-b)(s-c)}{s(s-a)} \\
 &= b^2 + c^2 + (b^2 + c^2 - a^2) - 2s(s-a) + 2(s-b)(s-c) \\
 &= (b-c)^2 + 2bc + 2bc \cos A + 2 \left(s^2 - s(2s-a) + bc - s(s-a) \right) \\
 &= (b-c)^2 + 2bc \cdot \frac{2s(s-a)}{bc} + 2(-2s^2 + 2as + bc)
 \end{aligned}$$

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$$= (b - c)^2 + 4s(s - a) - 4s(s - a) + 2bc = b^2 + c^2$$

$$\therefore n_a^2 + g_a^2 + 2rr_a = b^2 + c^2 \Rightarrow \frac{n_a^2 + g_a^2 + 2rr_a}{4R} = \frac{b^2 + c^2}{4R} \stackrel{\text{Tereshin}}{\leq} m_a \text{ (proved)}$$