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In ΔABC the following relationship holds:

$$\left(\sum r_a r_b\right)\left(\sum (r_a+r_b)^2(r_a+r_c)^2\right) \ge \left(\prod (r_a+r_b)^2\right)\left(\sum \cos^2\left(\frac{A}{2}\right)\right)$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum r_a r_b\right)\left(\sum (r_a+r_b)^2(r_a+r_c)^2\right) \overset{(1)}{\ge} \left(\prod (r_a+r_b)^2\right)\left(\sum \cos^2\left(\frac{A}{2}\right)\right)$$

$$r_b+r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2}} = \left(\frac{s}{\frac{s}{4R}}\right)\cos^2\frac{A}{2} = 4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c \overset{(a)}{=} 4R\cos^2\frac{A}{2} \text{ and analogs}$$

Now, LHS of (1) $= s^2\cdot\sum\left(r_a^2+\sum r_a r_b\right)^2 = s^2\sum\left(s^2\tan^2\frac{A}{2}+s^2\right)^2 \overset{(i)}{=} s^6\sum\sec^4\frac{A}{2}$

Using (a) and its analogs, RHS of (1)

$$= \prod\left(16R^2\cos^4\frac{A}{2}\right)\sum\cos^2\frac{A}{2}$$

$$= 16^3R^6\left(\frac{s}{4R}\right)^4\left(\sum\cos^2\frac{A}{2}\right) \overset{(ii)}{=} 16R^2s^4\left(\sum\cos^2\frac{A}{2}\right)$$

(i), (ii) $\Rightarrow$ *(1)* $\Leftrightarrow \left(\frac{s}{4R}\right)^2\sum\sec^4\frac{A}{2} \ge \sum\cos^2\frac{A}{2}$

$$\Leftrightarrow \left(\prod \cos^2\frac{A}{2}\right)\sum \sec^4\frac{A}{2} \geq \sum \cos^2\frac{A}{2} \Leftrightarrow \sum \sec^4\frac{A}{2} \geq \sum \sec^2\frac{B}{2}\sec^2\frac{C}{2}$$

$\rightarrow$ ***true*** $\because x^2 + y^2 + z^2 \geq xy + yz + zx$, ***where*** $x = \sec^2\frac{A}{2}, y = \sec^2\frac{B}{2}, z = \sec^2\frac{C}{2}$

(Proved)