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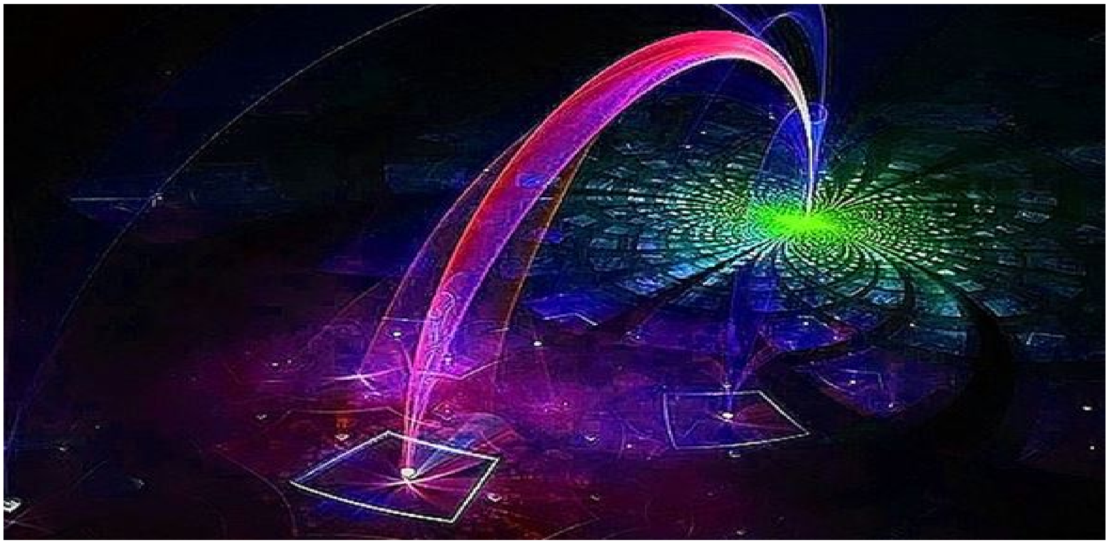

If $a,b>1,\frac{1}{a}+\frac{1}{b}=1$ then:

$$\min(a,b)\cdot\left(a^{a-1}+b^{b-1}\right)>\sqrt[2019]{ab}$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan

Solution 1 by Soumava Chakraborty-Kolkata-India, Solution 2 by Mokhtar Khassani-Mostaganem-Algerie

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\min(a,b)\cdot\left(a^{a-1}+b^{b-1}\right)\overset{(a)}{>}\sqrt[2019]{ab}.\text{ Let } f(x)=x^{x-1}\ \forall x>1$$

$$f''(x)=\frac{x^x}{x}\left(\ln x+\frac{x-1}{x}\right)^2+\frac{x^x}{x}\left(\frac{x+1}{x^2}\right)>0$$

$$\Rightarrow f(x)\text{ is convex} \therefore a^{a-1}+b^{b-1}\overset{Jensen}{\geq}2\left(\frac{a+b}{2}\right)^{\frac{a+b}{2}-1}$$

$$\Rightarrow a^{a-1}+b^{b-1}\overset{(1)}{\geq}2m^{m-1}\ \left(\text{where } m=\frac{a+b}{2}\right)$$

$$\text{Now, } 1=\frac{1}{a}+\frac{1}{b}\overset{Bergstom}{\geq}\frac{4}{a+b}\Rightarrow\frac{a+b}{2}\geq 2\Rightarrow m\geq 2\therefore m-1\geq 1$$

$$\therefore 2m^{m-1}=2\big(1+(m-1)\big)^{m-1}\overset{Bernoulii}{\geq}2\left(1+(m-1)^2\right)=2(m^2-2m+2)$$

$$\Rightarrow 2m^{m-1} \overset{(2)}{\gtreqless} 2(m^2-2m+2) \because a,b>1, \therefore \min(a,b) \overset{(3)}{\gtrless} 1$$

(3) along with (1).(2)⇒ LHS of (a) $\overset{(i)}{\gtrless} 2(m^2-2m+2)$

Again, $\because \frac{1}{a}+\frac{1}{b}=1, \therefore ab=a+b=2m$

$$\therefore (ab)^{\frac{1}{2019}}=(2m)^{\frac{1}{2019}}=\left(1+(2m-1)\right)^{\frac{1}{2019}} \overset{Bernoulli}{\lesseqgtr} 1+\frac{2m-1}{2019} \Rightarrow \textbf{\textit{RHS of (a)}} \overset{(ii)}{\lesseqgtr} 1+\frac{2m-1}{2019}$$

(i), (ii) ⇒ in order to prove (a), it suffices to prove: $2m^2-4m+4>1+\frac{2m-1}{2019}$

$$\Leftrightarrow 2019(2m^2-4m+3)>2m-1$$

$$\Leftrightarrow 4038m^2-8078m+6058>0 \Leftrightarrow 2019m^2-4039m+3029>0$$

$$\Leftrightarrow (m-2)(2019(m-2)+4037)+3027>0$$

$\rightarrow$ ***true*** $\because m-2\geq 0 \Rightarrow$ ***(a) is true (proved)***

Solution 2 by Mokhtar Khassani-Mostaganem-Algerie

Since inequality is symmetric we can assume that: $a\leq b$: $\min(a,b)=a$

Hölder $\overset{\frac{1}{a}+\frac{1}{b}=1}{\rightarrow} a^{a-1}+b^{b-1}\geq ab \Rightarrow a\left(a^{a-1}+b^{b-1}\right)\geq a^2b\geq ab>\sqrt[2019]{ab}$ ***(Proved)***