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If $a, b, c > 0, \frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}=\frac{17}{6}$ **then:**

$$\frac{abc+bc+2a}{2bc+a+1}+\frac{abc+ca+2b}{2ca+b+1}+\frac{abc+ab+2c}{2ab+c+1}>\frac{1}{3}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Șerban George Florin-Romania, Solution 2 by Remus Florin Stanca-Romania, Solution 3 by Ionuț Florin Voinea-Romania

Solution 1 by Șerban George Florin-Romania

$$\frac{abc+bc+2a}{2bc+a+1}=\frac{2a\cdot\left[\frac{(a+1)bc}{2a}+1\right]}{(a+1)\left(\frac{2bc}{a+1}+1\right)}\geq\frac{2a}{a+1}$$

$$\Rightarrow\frac{(a+1)bc}{2a}\geq\frac{2bc}{a+1}\Rightarrow(a+1)^2\geq4a, a^2-2a+1\geq0, (a+1)^2\geq0$$

true, $(\forall)a>0$

$$\frac{abc+ca+2b}{2ca+b+1}=\frac{2b\left[\frac{(b+1)ac}{2b}+1\right]}{(b+1)\left(\frac{2ca}{b+1}+1\right)}\geq\frac{2b}{b+1}$$

$$\Rightarrow\frac{(b+1)ac}{2b}\geq\frac{2ac}{b+1}\Rightarrow(b+1)^2\geq4b, (b-1)^2\geq0, \textit{true}$$

$(\forall)b>0$

$$\frac{abc+ab+2c}{2ab+c+1}=\frac{2c\left[\frac{(c+1)ab}{2c}+1\right]}{(c+1)\left(\frac{2ab}{c+1}+1\right)}\geq\frac{2c}{c+1}$$

$$\Rightarrow\frac{(c+1)ab}{2c}\geq\frac{2ab}{c+1}\Rightarrow(c+1)^2\geq 4c\Rightarrow(c-1)^2\geq 0$$

true, $(\forall)c>0$

$$\Rightarrow\sum_{cyc}\frac{abc+bc+2a}{2bc+a+1}\geq\sum_{cyc}\frac{2a}{a+1}=\sum_{cyc}\left(2-\frac{1}{a+1}\right)=$$

$$=6-2\sum_{cyc}\frac{1}{a+1}=6-2\cdot\frac{17}{6}=6-\frac{17}{3}=\frac{1}{3}, \textbf{\textit{true}}$$

Solution 2 by Remus Florin Stanca-Romania

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}=\frac{3\cdot 6-1}{6}=3-\frac{1}{6}\Rightarrow\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}=\frac{1}{6}$$

$$\Rightarrow\frac{1-a}{a+1}+\frac{1-b}{1+b}+\frac{1-c}{c+1}=\frac{17}{6}-\left(\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}\right)=\frac{17}{6}-\frac{1}{6}=\frac{16}{6}=\frac{8}{3}\Rightarrow$$

$$\Rightarrow\frac{(1-a)(bc+1)}{abc+a+bc+1}+\frac{(1-b)(ac+1)}{abc+b+ac+1}+\frac{(1-c)(ab+1)}{abc+c+ab+1}=\frac{8}{3}\ \textbf{\textit{(1)}}$$

the inequality can be written as:

$$\sum_{cyc}\frac{2bc+a+1-bc+a-1+abc}{2bc+a+1}>\frac{1}{3}\Leftrightarrow\sum_{cyc}\frac{bc+1-a-abc}{2bc+a+1}<\frac{8}{3}\overset{(1)}{\Leftrightarrow}$$

$$\overset{(1)}{\Leftrightarrow}\frac{(1-a)(bc+1)}{abc+a+bc+1}-\frac{(1-a)(bc+1)}{2bc+a+1}+\frac{(1-b)(ac+1)}{abc+b+ac+1}-$$
$$-\frac{(1-b)(ac+1)}{2ac+b+1}+\frac{(1-c)(ab+1)}{abc+c+ab+1}-\frac{(1-c)(ab+1)}{2ab+c+1}>0$$

$$\Leftrightarrow\sum_{cyc}\frac{(1-a)^2(bc+1)bc}{(2bc+a+1)(abc+a+bc+1)}>0\ \textbf{\textit{(true)}}\Rightarrow\frac{abc+bc+2a}{2bc+a+1}+\frac{abc+ac+2ab}{2ac+b+1}+\frac{abc+ab+2c}{2ab+c+1}>\frac{1}{3}$$

(Q.E.D.)

Solution 3 by Ionuț Florin Voinea-Romania

$$3=\frac{a+1}{a+1}+\frac{b+1}{b+1}+\frac{c+1}{c+1}=\left(\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}\right)+\frac{17}{6}\Rightarrow$$

$$\Rightarrow\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}=3-\frac{17}{6}=\frac{1}{6}\Rightarrow$$

$$\Rightarrow \frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}=\frac{1}{6}$$

$\frac{a}{a+1}\le\frac{1}{6}\Rightarrow 6a\le a+1\Rightarrow 0<a\le\frac{1}{5}$. ***In the same way:*** $0<b\le\frac{1}{5};\ 0<c\le\frac{1}{5}$

$$\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}=\frac{1}{6}\Big|\cdot 2\Leftrightarrow\frac{2a}{a+1}+\frac{2b}{b+1}+\frac{2c}{c+1}=\frac{1}{3}$$

$$\frac{abc+bc+2a}{2bc+a+1}>\frac{2a}{a+1}\Leftrightarrow a^2bc+abc+2a^2+abc+bc+2a>$$

$$>4abc+2a^2+2a\Leftrightarrow a^2bc-2abc+bc>0\Leftrightarrow$$

$bc(a-1)^2>0$ ***true because*** $a<\frac{1}{5}<1$

So, $\frac{abc+bc+2a}{2bc+a+1}+\frac{abc+ca+2b}{2ca+b+1}+\frac{abc+ab+2c}{2ab+c+1}>\frac{2a}{a+1}+\frac{2b}{b+1}+\frac{2c}{c+1}=\frac{1}{3}$