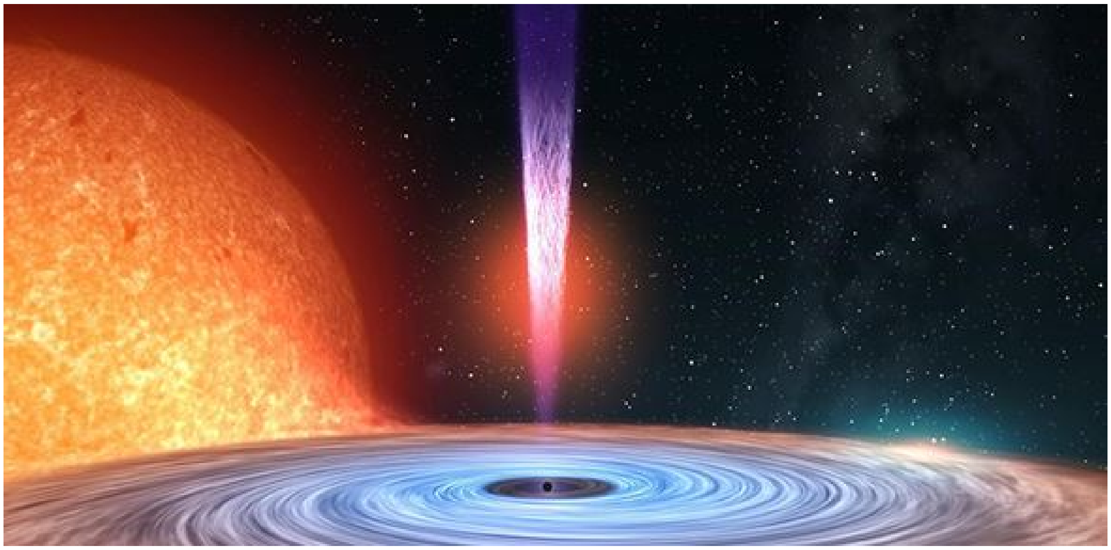

If $a, b, c > 1$ then:

$$\sum_{cyc} a^{\frac{3}{a}}\left(b^{\frac{1}{b}} + c^{\frac{1}{c}} - a^{\frac{1}{a}}\right) \le a^{\frac{1}{a}} \cdot b^{\frac{1}{b}} \cdot c^{\frac{1}{c}}\left(a^{\frac{1}{a}} + b^{\frac{1}{b}} + c^{\frac{1}{c}}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Șerban George Florin – Romania, Solution 2 by Adrian Popa – Romania

Solution 1 by Șerban George Florin – Romania

We donote $a^{\frac{1}{a}} = x, b^{\frac{1}{b}} = y, c^{\frac{1}{c}} = z, x, y, z > 1 \Rightarrow \sum x^3(y + z - x) \le xyz(x + y + z)$

Applying Schur's inequality

$$x^2(x-y)(x-z) + y^2(y-x)(y-z) + z^2(z-x)(z-y) \ge 0$$

$$\Rightarrow x^4 - x^3z - x^3y + x^2yz + y^4 - y^3z - xy^3 + xy^2z + z^4 - yz^3 - xz^3 + xyz^2 \ge 0 \Rightarrow$$

$$\Rightarrow \sum x^4 + xyz(x + y + z) - \sum(x^3y + x^3z) \ge 0 \Rightarrow$$

$$\Rightarrow xyz(x + y + z) \ge \sum(x^3y + x^3z - x^4)$$

$$\Rightarrow \sum x^3(y + z - x) \le xyz(x + y + z) \ \textit{True.}$$

Solution 2 by Adrian Popa – Romania

$$a, b, c > 1 \Rightarrow \sum a^{\frac{3}{a}}\left(b^{\frac{1}{b}} + c^{\frac{1}{c}} - a^{\frac{1}{a}}\right) \le a^{\frac{1}{a}} \cdot b^{\frac{1}{b}} \cdot c^{\frac{1}{c}}\left(a^{\frac{1}{a}} + b^{\frac{1}{b}} + c^{\frac{1}{c}}\right)$$

$$a > 1 \Rightarrow a^{\frac{1}{a}} > 1.\ \textit{Let } a^{\frac{1}{a}} = x > 1$$

$$b > 1 \Rightarrow b^{\frac{1}{b}} > 1.\ \textit{Let } b^{\frac{1}{b}} = y > 1$$

$$c > 1 \Rightarrow c^{\frac{1}{c}} > 1.\ \textit{Let } c^{\frac{1}{c}} = z > 1$$

We must prove that $\sum x^3\,(y + z - x) \le xyz(x + y + z) \Leftrightarrow$

$$\Leftrightarrow x^3y + x^3z - x^4 + y^3x + y^3z - y^4 + z^3x + z^3y - z^4 \le x^2yz + xy^2z + xyz^2 \Leftrightarrow$$

$$\Leftrightarrow x^4 + y^4 + z^4 - x^3y - x^3z - y^3x - y^3z - z^3x - z^3y + x^2yz + xy^2z + xyz^2 \ge 0 \Leftrightarrow$$

$$\Leftrightarrow x^2(x^2 - xy - xz + yz) + y^2(y^2 - xy - yz + xz) + z^2(z^2 - zx - zy + xy) \ge 0 \Leftrightarrow$$

$$\Leftrightarrow x^2(x - y)(x - z) + y^2(y - x)(y - z) + z^2(z - x)(z - y) \ge 0$$ ***– True***

Schur's inequality for $k = 2$.