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Prove that:

$$\sum_{n=1}^{\infty}\frac{\exp(-\pi n)}{n(n+1)(2n+1)(2n)!!}=$$

$$Ei\left(\frac{\exp(-\pi)}{2}\right)-2\exp\left(\frac{\pi}{2}\right)\sqrt{2\pi}\,\mathrm{erfi}\left(\frac{\exp\left(-\frac{\pi}{2}\right)}{\sqrt{2}}\right)-\gamma-\log\left(\frac{\exp(-\pi)}{2}\right)+$$

$$+2\exp\left(\frac{\exp(-\pi)}{2+\pi}\right)-2\exp(\pi)+3$$

γ: **Euler-Mascheroni constant;** Ei: **exponential integral**

erfi: **imaginary error function**

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$$\sum_{n=1}^{\infty}\frac{\exp(-\pi n)}{n(n+1)(2n+1)(2n)!!}\overset{(2n)!!=n!2^n}{\rightarrow}\sum_{n=1}^{\infty}\frac{\left(\frac{e^{-\pi}}{2}\right)^n}{n(n+1)(2n+1)n!}-\underbrace{\sum_{n=1}^{\infty}\frac{\left(\frac{e^{-\pi}}{2}\right)^n}{nn!}}_{S_1}+$$

$$+\underbrace{\sum_{n=1}^{\infty}\frac{\left(\frac{e^{-\pi}}{2}\right)^n}{(n+1)n!}}_{S_2}-4\underbrace{\sum_{n=1}^{\infty}\frac{\left(\frac{e^{-\pi}}{2}\right)^n}{(2n+1)n!}}_{S_3}\quad \textbf{(1)}$$

$$S_1-\sum_{n=1}^{\infty}\frac{\left(\frac{e^{-\pi}}{2}\right)^n}{nn!}\overset{(I)}{\rightarrow}E_t\left(\frac{e^{-\pi}}{2}\right)-\gamma-\log\left(\frac{e^{-\pi}}{2}\right)-E_i\left(\frac{\exp(-\pi)}{2}\right)-\gamma-\log\left(\frac{\exp(-\pi)}{2}\right)\quad \textbf{(2)}$$

Where (1) $E_i(x)\overset{def}{=}\gamma+\log(x)+\sum_{n=1}^{\infty}\frac{x^n}{n\cdot n!}$, E_i ***– exponential integral***

$$S_2=\sum_{n=1}^{\infty}\frac{\left(\frac{e^{-\pi}}{2}\right)^n}{(n+1)n!}=\sum_{n=1}^{\infty}\frac{\left(\frac{e^{-\pi}}{2}\right)^n}{(n+1)!}=\sum_{n=1}^{\infty}\frac{\left(\frac{e^{-\pi}}{2}\right)^n}{(n+1)!}-2e^{\pi}-1\overset{(II)}{\Rightarrow}\frac{e^{\left(\frac{e^{-\pi}}{2}\right)}}{\frac{e^{-\pi}}{2}}-2e^{\pi}-1=2\exp\left(\frac{\exp(-\pi)}{2}+\pi\right)-2\exp(\pi)-1\quad \textbf{(3)}$$

Where (II) $\frac{e^x}{x}\overset{def}{=}\sum_{n=0}^{\infty}\frac{x^n}{n!}\cdot\frac{1}{x}=\sum_{n=0}^{\infty}\frac{x^{n-1}}{n!}=\sum_{n=-1}^{\infty}\frac{x^n}{(n+1)!}$

$$S_3=4\sum_{n=1}^{\infty}\frac{\left(\frac{e^{-\pi}}{2}\right)^n}{(2n+1)n!}=4\cdot\sum_{n=0}^{\infty}\frac{\left(\sqrt{\frac{e^{-\pi}}{2}}\right)^{2n}}{(2n+1)n!}=4\cdot\sqrt{\frac{2}{e^{-\pi}}}\cdot\sum_{n=1}^{\infty}\frac{\left(\sqrt{\frac{e^{-\pi}}{2}}\right)^{2n+1}}{(2n+1)n!}=$$

$$= 4\sqrt{2e^{\pi}}\left[\sum_{n=0}^{\infty}\frac{\left(\sqrt{\frac{e^{-\pi}}{2}}\right)^{2n+1}}{(2n+1)n!} - \sqrt{\frac{e^{-\pi}}{2}}\right] = -4 + 4\sqrt{2e^{\pi}}\cdot\sum_{n=0}^{\infty}\frac{\left(\sqrt{\frac{e^{-\pi}}{2}}\right)^{2n+1}}{(2n+1)n!}$$

$$\overset{(III)}{\Rightarrow} -4 + 4\sqrt{2e^{\pi}}\cdot\frac{\sqrt{\pi}}{2}\operatorname{erfi}\left(\sqrt{\frac{e^{-\pi}}{2}}\right) = -4 + 2e^{\left(\frac{\pi}{2}\right)}\cdot\sqrt{2\pi}\cdot\operatorname{erfi}\left(\frac{e^{\left(\frac{-\pi}{2}\right)}}{\sqrt{2}}\right) = -4 + 2\exp\left(\frac{\pi}{2}\right)\sqrt{2\pi}\operatorname{erfi}\left(\frac{\exp\left(-\frac{\pi}{2}\right)}{\sqrt{2}}\right) \quad (4)$$

Where (III) $\operatorname{erfi}(x) \overset{def}{=} \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{x^{(2n+1)}}{(2n+1)\cdot n!}$, $\operatorname{erfi}(x)$ ***- imaginary function***

Rewriting (1) with (2), (3) and (4) we get:

$$\sum_{n=1}^{\infty}\frac{\exp(-\pi n)}{n(n+1)(2n+1)(2n)!!} = S_1 + S_2 - S_3 = E_i\left(\frac{\exp(-\pi)}{2}\right) - \gamma - \log\left(\frac{\exp(-\pi)}{2}\right) +$$

$$+2\exp\left(\frac{\exp(-\pi)}{2} + \pi\right) - 2\exp(\pi) - 1 + 4 - 2\exp\left(\frac{\pi}{2}\right)\sqrt{2\pi}\operatorname{erfi}\left(\frac{\exp\left(-\frac{\pi}{2}\right)}{\sqrt{2}}\right) =$$

$$= E_i\left(\frac{\exp(-\pi)}{2}\right) - 2\exp\left(\frac{\pi}{2}\right)\sqrt{2\pi}\operatorname{erfi}\left(\frac{\exp\left(-\frac{\pi}{2}\right)}{\sqrt{2}}\right) - \gamma - \log\left(\frac{\exp(-\pi)}{2}\right) +$$

$$+2\exp\left(\frac{\exp(-\pi)}{2} + \pi\right) - 2\exp(\pi) + e$$