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Let $\begin{cases} x_0 = 2020 \\ x_{n+1} = 2x_n - n^2 + 2n + 2019 \cdot 2020^n, n = 0, 1, 2, \dots \end{cases}$

Find:

$$\Omega = x_{2030}$$

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Using the formulas: $S_1 = x + 2^2x^2 + \dots + nx^n = \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$,

$$S_2 = x + 2^2x^2 + \dots + n^2x^n = \frac{n^2x^{n+3} + x^{n+2}(-2n^2 - 2n + 1) + (n+1)^2x^{n+1} - x^2 - x}{(x-1)^3}$$

We denote $\alpha = 2019 \cdot 2020$, $b_n = -n^2 + 2n + \alpha^n$, $x_{n+1} = 2x_n + b_n$

$x_1 = 2x_0 + b_0$, $x_2 = 2x_1 + b_1 = 2^2x_0 + 2b_0 + b_1$, $x_3 = x_2 + b_2 = 2^3x_0 + 2^2b_0 + x_n =$

$$= 2^nx_0 + \sum_{k=0}^{n-1} 2^{n-k-1} b_k = 2^nx_0 + \sum_{k=0}^{n-1} 2^{n-k-1}\left(-k^2 + 2k + \alpha^k\right)$$

$$x_n = 2^nx_0 + 2^{n-1} \cdot \sum_{k=0}^{n-1} \frac{-k^2 + 2k + \alpha^k}{2^k} =$$

$$= 2^nx_0 + 2^{n-1} \cdot \left[-\sum_{k=0}^{n-1} k^2\left(\frac{1}{2}\right)^k + 2\sum_{k=0}^{n-1} k \cdot \left(\frac{1}{2}\right)^k + \sum_{k=0}^{n-1}\left(\frac{\alpha}{2}\right)^k\right]$$

$$\Omega = x_{2030} = 2^{2030}x_0 + 2^{2029}\left[-\sum_{k=0}^{2029} k^2\left(\frac{1}{2}\right)^k + 2\sum_{k=0}^{2029} k\left(\frac{1}{2}\right)^k + \sum_{k=0}^{2029}\left(\frac{\alpha}{2}\right)^k\right]$$

$$\sum_{k=0}^{2029}\left(\frac{\alpha}{2}\right)^k = 1 + \left(\frac{\alpha}{2}\right) + \cdots \left(\frac{\alpha}{2}\right)^{2029} = \frac{\left(\frac{\alpha}{2}\right)^{2030} - 1}{\frac{\alpha}{2} - 1} = \frac{\alpha^{2030} - 2^{2030}}{2^{2029}(\alpha - 2)}$$

$$\sum_{k=0}^{2029} k\left(\frac{1}{2}\right)^k = \frac{1}{2} + 2\left(\frac{1}{2}\right)^2 + \cdots + 2029\left(\frac{1}{2}\right)^{2029} = \frac{2029\left(\frac{1}{2}\right)^{2031} - 2030\left(\frac{1}{2}\right)^{2030} + \frac{1}{2}}{\frac{1}{4}} =$$

$$= \frac{\frac{2029}{2^{2031}} - \frac{2030}{2^{2030}} + \frac{1}{2}}{\frac{1}{4}} = \frac{2029 - 4060 + 2^{2030}}{2^{2029}} = \frac{2^{2030} - 2031}{2^{2029}}$$

$$\sum_{k=0}^{2029} k^2\left(\frac{1}{2}\right)^k = \frac{2029\left(\frac{1}{2}\right)^{2032} + \left(-2\cdot 2029^2 - 2\cdot 2029 + 1\right)\left(\frac{1}{2}\right)^{2031} + 2030^2\left(\frac{1}{2}\right)^{2030} - \frac{3}{4}}{-\frac{1}{8}} =$$

$$= \left(\frac{2029}{2^{2032}} - \frac{8237739}{2^{2031}} + \frac{4120900}{2^{2030}} - \frac{3}{4}\right)\cdot\left(-\frac{8}{1}\right) =$$

$$= \frac{2029 - 16475478 + 16483600 - 3\cdot 2^{2030}}{2^{2032}}\cdot\left(-\frac{8}{1}\right) = \frac{3\cdot 2^{2030} - 10\cdot 151}{2^{2029}}$$

$$\Omega = 2^{2030}\cdot 2020 + 2^{2029}\left[\frac{3\cdot 2^{2030} - 10151}{2^{2029}} + \frac{2^{2030} - 2031}{2^{2028}} + \frac{\alpha^{2030} - 2^{2030}}{2^{2029}(\alpha - 2)}\right] =$$

$$= 2^{2030}\cdot 2020 - 3\cdot 2^{2030} + 10151 + 2^{2031} - 4062 + \frac{\alpha^{2030} - 2^{2030}}{\alpha - 2} =$$

$$= 2^{2030}(2020 - 3 + 2) + 6089 + \frac{\alpha^{2030} - 2^{2030}}{\alpha - 2} =$$

$$\Omega = 2019\cdot 2^{2030} + 6089 + \frac{20192020^{2030} - 2^{2030}}{20192018}$$

$$\Omega = \frac{40767684341\cdot 2^{2030} + 122949197602 + 20192020^{2030}}{20192018}$$