PROPOSED PROBLEM

PEDRO H.O. PANTOJA
NATAL/RN, BRAZIL

Evaluate:

$$\int_0^{\frac{\pi}{4}} \cos^2(x)\cdot\ln(1+\cos(4x))dx$$

Solution 1 by Kamel Benaicha - University Algiers - Algerie.

$$\Omega = \int_0^{\frac{\pi}{4}} \cos^2(x)\ln(1+\cos(ux))dx$$

$$\text{Let } I = \int_0^{\frac{\pi}{4}} \sin^2(x)\ln(1+\cos(ux))dx$$

$$\text{Then: } \Omega + I = \int_0^{\frac{\pi}{4}} \ln(1+\cos(ux))dx = \int_0^{\frac{\pi}{4}} \ln(2\cos^2(2x))dx$$

$$= \int_0^{\frac{\pi}{4}} \ln(2)dx + \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\cos^2(t))dt \quad (t=2x)$$

$$= \frac{\pi}{4}\ln(2) + \int_0^{\frac{\pi}{2}} \ln(\cos(t))dt = \frac{\pi}{4}\ln(2) - \frac{\pi}{2}\ln(2)$$

$$= -\frac{\pi}{4}\ln(2) \tag{1}$$

$$\text{And: } \Omega = I = \int_0^{\frac{\pi}{4}} \cos(2x)\ln(1+\cos(ux))dx$$

$$= \int_0^{\frac{\pi}{4}} \cos(2x)\ln(2\cos^2(2x))dx$$

$$\therefore \Omega = I = \int_0^{\frac{\pi}{4}} \ln(2)\cos(2x)dx + \int_0^{\frac{\pi}{4}} \cos(2x)\ln(\cos^2(2x))dx$$

$$= \frac{\ln(2)}{2}\sin(2x)\Big|_0^{\frac{\pi}{4}} + \frac{1}{2}\int_0^1 \ln(1-t^2)dt \quad (t=\sin(2t))$$

$$= \frac{\ln(2)}{2} + \frac{1}{2}\ln(t)dt + \frac{1}{2}\int_0^1 \ln(1+t)dt$$

$$= \frac{\ln(2)}{2} + \frac{1}{2}(t\ln(t) + (1+t)\ln(t) - 2t)\Big|_0^1$$

$$= \frac{\ln(2)}{2} + \frac{1}{2}(2\ln(2)-2) = \frac{3}{2}\ln(2) - 1 \tag{2}$$

$$\text{So: } (1)+(2) \Leftrightarrow 2\Omega = -\frac{\pi}{4}\ln(2) + \frac{3}{2}\ln(2) - 1$$

$$\therefore \Omega = -\frac{\pi}{8}\ln(2) + \frac{3}{4}\ln(2) - \frac{1}{2}$$

$$\therefore \int_0^{\frac{\pi}{4}} \cos^2(x)\ln(1+\cos(ux))dx = \frac{(6-\pi)}{8}\ln(2) - \frac{1}{2}$$

□

Solution 2 by Remus Florin Stanca - Romania.

$$\cos(4x) = 2\cos^2(2x) - 1 \Rightarrow 1+\cos(4x) = 2\cos^2(2x) \Rightarrow I = \int_0^{\frac{\pi}{4}} \cos^2 x \ln(2\cos^2 2x)dx =$$

$$= \int_0^{\frac{\pi}{4}} \cos^2 x(\ln(2)+2\ln(\cos 2x))dx = \ln(2)\int_0^{\frac{\pi}{4}} \cos^2 x dx + 2\int_0^{\frac{\pi}{4}} \cos^2 x \ln(\cos 2x)dx =$$

$$= \ln(2)\int_0^{\frac{\pi}{4}} \cos^2 x dx + \int_0^{\frac{\pi}{4}} (2\cos^2 x - 1 + 1)\ln(\cos 2x)dx =$$

$$= \ln(2)\int_0^{\frac{\pi}{4}} \cos^2 x dx + \int_0^{\frac{\pi}{4}} \cos 2x \ln(\cos 2x)dx + \int_0^{\frac{\pi}{4}} \ln(\cos 2x)dx =$$

(a) $$= \ln(2)\cdot J + K + L$$

where $J = \int_0^{\frac{\pi}{4}} \cos^2 x dx, K = \int_0^{\frac{\pi}{4}} \cos 2x \ln(\cos 2x)dx, L = \int_0^{\frac{\pi}{4}} \ln(\cos 2x)dx$

(1)

$$J = \int_0^{\frac{\pi}{4}} \cos^2 x dx = \int_0^{\frac{\pi}{4}} \frac{2\cos^2 x - 1 + 1}{2}dx = \int_0^{\frac{\pi}{4}} \frac{\cos 2x}{2}dx + \int_0^{\frac{\pi}{4}} \frac{1}{2}dx = \frac{\pi}{8} + \frac{1}{2} = \frac{\pi+2}{8}$$

$$K = \int_0^{\frac{\pi}{4}} \cos 2x \ln(\cos 2x)dx, 2x = t \Rightarrow dx = \frac{dt}{2} \Rightarrow K = \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos x \ln(\cos x)dx =$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos x \ln(\sqrt{1-\sin^2 x})dx, \sin x = y \Rightarrow \cos x dx = dy \Rightarrow$$

(2) $$\Rightarrow K = \frac{1}{2}\ln(\sqrt{1-y^2})dy = \frac{1}{4}\int_0^1 \ln(1-y^2)dy$$

$$\int (1-y^2)dy = \int \ln(1-y^2)y' dy = y\ln(1-y^2) - \int \frac{y}{1-y^2}\cdot(-2y)dy =$$

$$= y\ln(1-y^2) + 2\int \frac{y^2}{1-y^2}dy = y\ln(1-y^2) - 2\int \frac{y^2-1+1}{y^2-1}dy =$$

$$= y\ln(1-y^2) - 2\int dy - \int \frac{y+1-(y-1)}{(y-1)(y+1)}dy = y\ln(1-y^2) - \int \frac{1}{y-1}dy + \int \frac{1}{y+1}dy - 2y =$$

$$= y\ln(1-y^2) - 2y - \ln(1-y) + \ln(y+1) + C \overset{(2)}{\Rightarrow}$$

$$\overset{(2)}{\Rightarrow} K = \frac{1}{4}\Big(\lim_{y\to 1^-}(y\ln(1-y^2) - \ln(1-y)) + \ln(2) - 2\Big) =$$

$$= \frac{1}{4}\Big(\ln(2) - 2 + \lim_{y\to 1_-}(y\ln(1-y) + y\ln(y+1) - \ln(1-y))\Big) =$$

$$= \frac{1}{4}\Big(\ln 2 - 2 + \ln(2) + \lim_{x\to 0_+} x\ln(x)\Big) =$$

(3) $$= \frac{1}{4}\left(2\ln(2) - 2 + \lim_{x\to 0_+}\frac{\ln(x)}{\frac{1}{x}}\right) \overset{\text{L'H}}{\underset{\frac{\infty}{\infty}}{=}} \frac{1}{4}\left(2\ln(2) - 2 + \lim_{x\to 0_+}\frac{1\frac{1}{x}}{-\frac{1}{x^2}}\right) = \frac{1}{4}(2\ln(2) - 2)$$

$$L=\int_0^{\frac{\pi}{4}}\ln(\cos 2x)dx, 2x=t\Rightarrow dx=\frac{dt}{2}\Rightarrow L=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln(\cos x), x=\frac{\pi}{2}-t\Rightarrow dx=-dt\Rightarrow$$

$$\Rightarrow L=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln(\sin x)dx\Rightarrow 2L=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln\left(\frac{\sin 2x}{2}\right)dx\Rightarrow$$

$$\Rightarrow L=\frac{1}{4}\int_0^{\frac{\pi}{2}}\ln(\sin 2x)dx-\frac{\pi\ln(2)}{8}, 2x=t\Rightarrow 2dx=dt\Rightarrow L=\frac{1}{8}\int_0^{\pi}\ln(\sin x)dx-\frac{x\ln(2)}{8}=$$

$$=\frac{1}{8}\int_0^{\frac{\pi}{2}}\ln(\sin(x))dx+\frac{1}{8}\int_{\frac{\pi}{2}}^{\pi}\ln(\sin x)dx-\frac{\pi\ln(2)}{8}=\frac{1}{4}\int_0^{\frac{\pi}{2}}\ln(\sin x)dx-\frac{\pi\ln(2)}{8}\Rightarrow$$

$$\Rightarrow\frac{1}{4}\int_0^{\frac{\pi}{2}}\ln(\sin(x))dx=-\frac{\pi\ln(2)}{8}\Rightarrow\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln(\sin x)dx=\frac{-\pi\ln 2}{4}\Rightarrow$$

$$\Rightarrow L=\frac{-\pi\ln(2)}{4}\tag{4}$$

$$\underset{(1);(2);(3);(4);(a)}{\Rightarrow} I=\frac{\pi\ln(2)+2\ln(2)}{8}+\frac{4\ln(2)-4}{8}-\frac{2\pi\ln(2)}{8}=$$

$$=\frac{-\pi\ln(2)+6\ln(2)-4}{8}=\frac{(6-\pi)\ln(2)-4}{8}\Rightarrow$$

$$\Rightarrow I=\frac{(6-\pi)\ln(2)-4}{8}$$

□

Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania
Email address: dansitaru63@yahoo.com