PROPOSED PROBLEM

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If $a, b, c \in [0,1)$, then:

$$8\int_0^a\left(\int_0^b\left(\int_0^c \frac{\sin^{-1}x\cdot\sin^{-1}y\cdot\sin^{-1}z}{(1+\sin^{-1}x)(1+\sin^{-1}y)(1+\sin^{-1}z)}dz\right)dy\right)dx \le a^2b^2c^2$$

Solution 1 by Soumava Chakraborty-"SoftWeb Technology"-Kolkata-India.

$$8\int_0^a\left(\int_0^b\left(\int_0^c \frac{\sin^{-1}x\cdot\sin^{-1}y\cdot\sin^{-1}z}{(1+\sin^{-1}x)(1+\sin^{-1}y)(1+\sin^{-1}z)}dz\right)dy\right)dx \overset{(m)}{\le} a^2b^2c^2$$

$$0\le z\le c, 0\le y\le b, 0\le x\le a \text{ and } \because a,b,c<1 \text{ as } a,b,c\in[0,1)$$

$$\therefore 0\le x,y,z<1 \Rightarrow 0\le \sin^{-1}x, \sin^{-1}y, \sin^{-1}z \overset{(a)}{<} \frac{\pi}{2}$$

Let $f(\theta)=\theta\sin\theta+\sin\theta-\theta \quad \forall\theta\in\left[0,\frac{\pi}{2}\right)$

$$f'(\theta)=\theta\cos\theta+\sin\theta+\cos\theta-1\overset{?}{\ge}0$$

$$\Leftrightarrow \theta\cos\theta+\sin\theta \underset{(1)}{\overset{?}{\ge}} 2\sin^2\frac{\theta}{2}$$

$$\because \theta\ge\sin\theta \quad \forall\theta\in\left[0,\frac{\pi}{2}\right)\therefore \text{ LHS of (1)}$$

$$\ge \sin\theta(1+\cos\theta)=2\sin\theta\cos^2\frac{\theta}{2}\overset{?}{\ge}2\sin^2\frac{\theta}{2}$$

$$\Leftrightarrow \sin\theta\overset{?}{\ge}\tan^2\frac{\theta}{2}\Leftrightarrow\frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\overset{?}{\ge}\tan^2\frac{\theta}{2}$$

$$\Leftrightarrow \tan^4\frac{\theta}{2}+\tan^2\frac{\theta}{2}-2\tan\frac{\theta}{2}\overset{?}{\le}0$$

$$\Leftrightarrow t(t^3+t-2)\overset{?}{\le}0\Big(\text{ where } t=\tan\frac{\theta}{2}\Big)$$

$$\Leftrightarrow t(t-1)(t^2+t+2)\overset{?}{\le}0\rightarrow \text{ true } \because 0\le t<1 \text{ as } 0\le\frac{\theta}{2}<\frac{\pi}{4}$$

$$\Rightarrow \text{(1) is true } \Rightarrow f'(\theta)\ge0 \quad \forall\theta\in\left[0,\frac{\pi}{2}\right)$$

$$\Rightarrow \forall\theta\in\left[0,\frac{\pi}{2}\right), f(\theta) \text{ is } \uparrow \text{ and } \because f(0)=0$$

$$\therefore \forall\theta\in\left[0,\frac{\pi}{2}\right), \theta\sin\theta+\sin\theta-\theta\ge0$$

$$\Rightarrow \sin\theta(\theta+1)\ge\theta\Rightarrow\frac{\theta}{\theta+1}\overset{(i)}{\le}\sin\theta$$

$$\text{(i) along with (a)} \Rightarrow \frac{\sin^{-1}x}{1+\sin^{-1}x}\le x \text{ and analogs}$$

$$\text{(ii) and analogs } \Rightarrow \text{LHS of (m)}$$

$$\leq 8\int_0^a\left(\int_0^b\left(\int_0^c xyzdz\right)dy\right)dx$$
$$= 8\int_0^a\left(\frac{xc^2}{2}\left(\int_0^b ydy\right)\right)dx$$
$$= 8\cdot\frac{c^2b^2}{4}\int_0^a xdx = \frac{8a^2b^2c^2}{8} = a^2b^2c^2$$
$$\Rightarrow \text{ (m) is true (Proved)}$$

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Solution 2 by Soumitra Mandal-Chandar Nagore-India.
Let $f(x) = x + x\sin^{-1}x - \sin^{-1}x$ for all $x \geq 0$

$$f'(x) = 1 + \sin^{-1}x + \frac{x}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 1 + \sin^{-1}x - \sqrt{\frac{1-x}{1+x}}$$
$$f''(x) = \frac{1}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}} + \frac{\sqrt{1-x}}{2\sqrt{(1+x)^3}} = \frac{2-x-x^2}{\sqrt{(1-x^2)^3}}$$
$$\therefore \text{ for } f'(c) = 0, \text{ where } c \text{ is critical point}$$

$1+\sin^{-1}c - \sqrt{\dfrac{1-c}{1+c}} = 0 \Rightarrow c = 0 \therefore f''(c) = \dfrac{2-c-c^2}{\sqrt{(1-c^2)^3}} = 2 > 0$ hence f has local

$$\text{minimum at } c = 0 \text{ then } f(x) \geq f(c) = f(0) = 0 \Rightarrow x \geq \frac{\sin^{-1}x}{1+\sin^{-1}x}$$

$$\therefore 8\int_0^a\int_0^b\int_0^c \frac{\sin^{-1}x\cdot\sin^{-1}y\cdot\sin^{-1}z}{(1+\sin^{-1}x)(1+\sin^{-1}y)(1+\sin^{-1}z)}dxdydz \leq 8\left(\int_0^a xdx\right)\left(\int_0^b ydy\right)\left(\int_0^c zdz\right)$$
$$= a^2b^2c^2 \text{ (proved)}$$

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