PROPOSED PROBLEM

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Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $4^{-(x+y)} \leq \frac{f(x)f(y)}{(x^4+1)(y^4+1)} \leq \frac{f(x+y)}{(x+y)^4+1}$ for all $x, y \in \mathbb{R}$.

Solution by Marian Ursărescu - National College "Roman Vodă" - Roman - Romania.

Let $y = x \Rightarrow 0 < 4^{-2x} \leq \frac{(f(x))^2}{(x^4+1)^2} \leq \frac{f(2x)}{(2x)^4+1} \Rightarrow$

$$f(2x) > 0, \forall x \in \mathbb{R} \Rightarrow f(x) > 0, \forall x \in \mathbb{R} \tag{1}$$

$$\text{Let } y = x = 0 \Rightarrow 1 \leq (f(0))^2 \leq f(0) \tag{2}$$

$$\text{From } (1) + (2) \Rightarrow f(0) = 1 \tag{3}$$

Let $y = 0 \Rightarrow 4^{-x} \leq \frac{f(x) \cdot f(0)}{(x^4+1)} \leq \frac{f(x)}{x^4+1} \overset{(3)}{\Rightarrow}$

$$f(x) \geq (x^4+1)4^{-x} \tag{4}$$

Let $y = -x \Rightarrow 1 \leq \frac{f(x) \cdot f(-x)}{(x^4+1)^2} \leq \frac{f(0)}{1} \overset{(3)}{\Rightarrow}$

$$\frac{f(x) \cdot f(-x)}{(x^4+1)^2} = 1 \overset{(1)}{\Rightarrow} f(-x) = \frac{(x^4+1)^2}{f(x)} \tag{5}$$

In relation (4) $\quad x \to -x \Rightarrow f(-x) \geq (x^4+1)4^{x} \overset{(5)}{\Rightarrow}$

$$\frac{(x^4+1)^2}{f(x)} \geq (x^4+1)4^{x} \overset{(1)}{\Rightarrow} f(x) \leq (x^4+1)4^{-x} \tag{6}$$

From $(4) + (6) \Rightarrow f(x) = (x^4+1)4^{-x}$ □

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