RMM - Calculus Marathon 701 - 800

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701. Prove the following relation:

$$\int_0^\infty \left(1-\frac{1}{\sqrt{x}}-\frac{1}{x}\right)\left(\pi\cot(\pi x)-\frac{1}{x}\right)\sin(\pi x)\,e^{-\pi x}dx=\frac{1}{4}\left(\pi\left(2\sqrt{5\sqrt{2}-7}-\pi+3\right)+2\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Tobi Joshua-Nigeria

$$I=\int_0^\infty \left(1-\frac{1}{\sqrt{x}}-\frac{1}{x}\right)\left(\pi\cot\pi x-\frac{1}{x}\right)\sin\pi x\, e^{-\pi x}dx$$

$$I=\int_0^\infty \left(1-\frac{1}{\sqrt{x}}-\frac{1}{x}\right)\left(\pi\cos\pi x\, e^{-\pi x}-\frac{\sin\pi x\ e^{-\pi x}}{x}\right)dx$$

$$I=\int_0^\infty \left(1-\frac{1}{\sqrt{x}}-\frac{1}{x}\right)(\pi\cos\pi x\, e^{-\pi x})-\left(\frac{\sin\pi x\, e^{-\pi x}}{x}\right)dx+\int_0^\infty\left(\frac{1}{\sqrt{x}}+\frac{1}{x}\right)\left(\frac{\sin\pi x e^{-\pi x}}{x}\right)dx$$

$$I=\int_0^\infty \left(1-\frac{1}{\sqrt{x}}-\frac{1}{x}\right)(\pi\cos\pi x\, e^{-\pi x})-\int_0^\infty\left(\frac{\sin\pi x\, e^{-\pi x}}{x}\right)dx+\int_0^\infty\left(\frac{\sin\pi x\, e^{-\pi x}}{x\sqrt{x}}\right)dx+$$

$$+\int_0^\infty\left(\frac{\sin\pi x e^{-\pi x}}{x^2}\right)dx \Rightarrow \textit{applying by part}$$

$$I=\int_0^\infty \left(1-\frac{1}{\sqrt{x}}\right)(\pi\cos\pi x\, e^{-\pi x})dx-\int_0^\infty\left(\frac{(\pi\cos\pi x\, e^{-\pi x})}{x}\right)dx-\int_0^\infty\left(\frac{\sin\pi x\, e^{-\pi x}}{x}\right)dx+$$

$$+\int_0^\infty\left(\frac{\sin\pi x\, e^{-\pi x}}{x\sqrt{x}}\right)dx+\frac{\sin\pi x\, e^{-\pi x}}{x}\Bigg|_0^\infty-\pi\int_0^\infty\left(\frac{\sin\pi x\, e^{-\pi x}}{x}\right)dx+\int_0^\infty\left(\frac{(\pi\cos\pi x\, e^{-\pi x})}{x}\right)dx$$

$$I=\int_0^\infty \left(1-\frac{1}{\sqrt{x}}\right)(\pi\cos\pi x\, e^{-\pi x})dx-(1+\pi)\int_0^\infty\left(\frac{\sin\pi x\, e^{-\pi x}}{x}\right)dx+\int_0^\infty\left(\frac{\sin\pi x\, e^{-\pi x}}{x\sqrt{x}}\right)dx+\pi$$

$$A=\int_0^\infty \left((\pi\cos\pi x\, e^{-\pi x})-\frac{(\pi\cos\pi x\, e^{-\pi x})}{\sqrt{x}}\right)dx-\int_0^\infty(1+\pi)\left(\frac{\sin\pi x\, e^{-\pi x}}{x}\right)dx+$$

$$+\int_0^\infty\left(\frac{\sin\pi x\, e^{-\pi x}}{x\sqrt{x}}\right)dx+\pi$$

$$A=\frac{1}{2}-Re\,\pi\int_0^\infty\frac{\left(e^{-\pi x(i+1)}\right)}{\sqrt{x}}dx-(1+\pi)\,L\left\{\frac{\sin\pi x}{x}\right\}_{s=\pi}+Im\int_0^\infty\frac{\left(e^{-\pi x(-i+1)}\right)}{x\sqrt{x}}dx+\pi$$

$$A=\frac{1}{2}-Re\frac{\sqrt{\pi}}{\sqrt{i+1}}\int_0^\infty x^{\frac{-1}{2}}\left(e^{-x}\right)dx-(1+\pi)\left(\frac{\pi}{4}\right)+$$

$$+Im\left(\pi^{\frac{3}{2}}\right)\sqrt{(1-i)}\int_0^\infty x^{-\frac{3}{2}}\left(e^{-x}\right)dx+\pi$$

$$A=\frac{1}{2}-Re\frac{\sqrt{\pi}}{\sqrt{i+1}}\Gamma\left(\frac{1}{2}\right)-(1+\pi)\left(\frac{\pi}{4}\right)+Im\left(\pi^{\frac{1}{2}}\right)\sqrt{(1-i)}\Gamma\left(-\frac{1}{2}\right)+\pi$$

$$A=\frac{1}{2}-Re\frac{\pi}{\sqrt{i+1}}-(1+\pi)\left(\frac{\pi}{4}\right)-2Im(\pi)\sqrt{(1-i)}+\pi$$

$$A=\frac{1}{2}-\pi Re\frac{1}{\left(\sqrt{\sqrt{2}}\right)}\left[\cos\left(\frac{\pi}{8}\right)-i\sin\left(\frac{\pi}{8}\right)\right]-(1+\pi)\left(\frac{\pi}{4}\right)-$$

$$-2Im(\pi)\left(\sqrt{\sqrt{2}}\right)\left[\cos\left(\frac{\pi}{8}\right)-i\sin\left(\frac{\pi}{8}\right)\right]+\pi$$

$$A=\frac{1}{2}-\pi\frac{1}{\left(\sqrt{\sqrt{2}}\right)}\left[\frac{\sqrt{2+\sqrt{2}}}{2}\right]-(1+\pi)\left(\frac{\pi}{4}\right)+2(\pi)\left(\sqrt{\sqrt{2}}\right)\left[\frac{\sqrt{2-\sqrt{2}}}{2}\right]+\pi$$

$$A=\frac{1}{2}-\frac{\pi\sqrt{1+\sqrt{2}}}{2}+\pi\sqrt{2\sqrt{2}-2}-\left(\frac{\pi^2+\pi}{4}\right)+\pi$$

$$A=\frac{1}{2}+\frac{\pi}{2}\left[2\sqrt{2\sqrt{2}-2}-\sqrt{1+\sqrt{2}}\right]-\left(\frac{\pi^2+\pi}{4}\right)+\pi$$

$$I=\frac{1}{2}+\frac{\pi}{2}\left[\sqrt{5\sqrt{2}-7}\right]-\left(\frac{\pi^2+3\pi}{4}\right)$$

$$I=\frac{1}{4}\left(2+2\pi\left[\sqrt{5\sqrt{2}-7}\right]-(\pi^2+3\pi)\right)$$

702. If:

$$\Omega(\theta)=\left(\sum_{k=0}^{n}\left(2^k\tan\left(2^k\theta\right)\right)\right)+2^{n+1}\cot\left(2^{n+1}\theta\right)$$

then:

$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{120}}\left(\left(\Omega(\theta)\right)^2+\left(\Omega\left(\frac{\theta}{2}\right)\right)^{-2}\right)d\theta$$

$$=\frac{5\pi+12}{12}+\frac{4\left(\sqrt{8-2\sqrt{6}-2\sqrt{2}}-\sqrt{2-\sqrt{3}}\right)}{4-\sqrt{6}-\sqrt{2}}+\frac{3\left(1-\sqrt{6}+\sqrt{12}\right)}{\sqrt{3}-\sqrt{6}}$$

Proposed by Naren Bhandari-Bajura-Nepal

Solution by Mokhtar Khassani-Mostaganem-Algerie

$$\Omega(\theta)=\sum_{k=0}^{n}2^k\tan\left(\theta 2^k\right)+2^{k+1}\cot\left(\theta 2^{k+1}\right)$$

$$\therefore\prod_{k=0}^{n}\cos\left(\theta 2^k\right)=\prod_{k=0}^{n}\frac{\sin\left(\theta 2^{k+1}\right)}{2\sin\left(\theta 2^k\right)}=\frac{\sin\left(\theta 2^{k+1}\right)}{2^{n+1}\sin\theta}\Rightarrow\sum_{k=0}^{n}\log\left(\cos\left(\theta 2^k\right)\right)=\log\frac{\sin\left(\theta 2^{k+1}\right)}{2^{n+1}\sin\theta}$$

$$\Rightarrow\sum_{k=0}^{n}\frac{d\log\left(\cos\left(\theta 2^k\right)\right)}{d\theta}=\frac{d\log\left(\frac{\sin\left(\theta 2^{k+1}\right)}{2^{n+1}\sin\theta}\right)}{d\theta}\Rightarrow\sum_{k=0}^{n}2^k\tan\left(\theta 2^k\right)=$$

$$=\cot\theta-2^{n+1}\cot\left(\theta 2^{n+1}\right)$$

$$\Rightarrow\Omega(\theta)=\cot\theta\Rightarrow M=\int_{\frac{\pi}{4}}^{\frac{5\pi}{120}}\left(\Omega^2(\theta)+\frac{1}{\Omega^2\left(\frac{\theta}{2}\right)}\right)d\theta=\int_{\frac{\pi}{4}}^{\frac{5\pi}{120}}\left(\cot^2\theta+\frac{1}{\cot^2\frac{\theta}{2}}\right)d\theta$$

$$=\{-\theta-\cot\theta\}_{\frac{\pi}{4}}^{\frac{5\pi}{120}}+\left\{-\theta+2\tan\frac{\theta}{2}\right\}_{\frac{\pi}{4}}^{\frac{5\pi}{120}}=-\frac{5\pi}{60}-\cot\frac{5\pi}{120}+2\tan\frac{5\pi}{240}+\frac{\pi}{2}+1-2\tan\frac{\pi}{8}$$

$$=\frac{5\pi}{12}+3-2\sqrt{2}-\cot\frac{\pi}{24}+2\tan\frac{\pi}{48}\quad\left\{\tan\frac{\pi}{8}=\frac{\sin\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}=\sqrt{2}-1\right\}$$

$$\cot\frac{\pi}{24}=\frac{1+\cos\frac{\pi}{12}}{\sin\frac{\pi}{12}}=\frac{1+\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}=$$

$$=2+\sqrt{2}+\sqrt{3}+\sqrt{6}\left\{\cos\frac{\pi}{12}=\sqrt{\frac{1+\cos\frac{\pi}{6}}{2}}=\frac{\sqrt{3}+1}{2\sqrt{2}}\right\}$$

$$\tan\frac{\pi}{48}=\frac{\sin\frac{\pi}{24}}{1+\cos\frac{\pi}{24}}=\frac{\sqrt{\frac{1-\cos\frac{\pi}{12}}{2}}}{1+\sqrt{\frac{1+\cos\frac{\pi}{12}}{2}}}=\frac{\sqrt{2-\sqrt{2+\sqrt{3}}}}{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}$$

$$M=\frac{5\pi}{12}+1-3\sqrt{2}-\sqrt{3}-\sqrt{6}+2\frac{\sqrt{2-\sqrt{2+\sqrt{3}}}}{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}$$

$$=\frac{5\pi}{12}+1+2\frac{\frac{\sqrt{2}}{2}\sqrt{4-\sqrt{6}-\sqrt{2}}}{2+\frac{\sqrt{2}}{2}\sqrt{4+\sqrt{6}+\sqrt{2}}}-3\sqrt{2}-\sqrt{3}-\sqrt{6}$$

$$=\frac{5\pi}{12}+1+\frac{\sqrt{2}\sqrt{4-\sqrt{6}-\sqrt{2}}\left(2-\frac{\sqrt{2}}{2}\sqrt{4+\sqrt{6}+\sqrt{2}}\right)}{\frac{4-\sqrt{6}-\sqrt{2}}{2}}-3\sqrt{2}-\sqrt{3}-\sqrt{6}$$

$$=\frac{5\pi}{12}+1+\frac{2\sqrt{8-2\sqrt{6}-2\sqrt{2}}-2\sqrt{2-\sqrt{3}}}{\frac{4-\sqrt{6}-\sqrt{2}}{2}}-3\sqrt{2}+\frac{3}{\sqrt{3}-\sqrt{6}}$$

$$=\frac{5\pi}{12}+1+4\frac{\sqrt{8-2\sqrt{6}-2\sqrt{2}}-\sqrt{2-\sqrt{3}}}{4-\sqrt{6}-\sqrt{2}}+3\frac{1-\sqrt{6}+\sqrt{12}}{\sqrt{3}-\sqrt{6}}$$

703. Prove that:

$$-\frac{1}{4}\int_0^\infty \frac{\ln(x)}{x^2+1}dx-\frac{1}{2}\int_1^2 \frac{\ln(x)}{x}dx-\frac{1}{4}\int_0^1 \frac{\ln^2(1+x^2)}{x^2}dx+\int_0^1 \frac{x\tan^{-1}x}{(1+x^2)}dx+$$

$$+\frac{1}{2}\int_0^1 \frac{\ln(1+x^2)}{1+x^2}dx+\frac{1}{8}\int_0^\infty \frac{\ln(x^2+1)}{x^2+1}dx=G-\frac{\pi}{4}\ln(2)$$

where G is Catalan's constant

Proposed by Abdul Hafeez Ayinde-Nigeria

Solution by Avishek Mitra-West Bengal-India

$$\Leftrightarrow I_1=\int_0^\infty \frac{\log x}{(1+x^2)}dx\Rightarrow x=\tan z\Rightarrow dx=\sec^2 z\,dz$$

$$=\int_0^{\frac{\pi}{2}}\frac{\log(\tan z)}{(1+\tan^2 z)}\cdot\sec^2 z\,dz=\int_0^{\frac{\pi}{2}}\log\left(\tan\left(\frac{\pi}{2}-z\right)\right)dz=\int_0^{\frac{\pi}{2}}\log(\cot z)\,dz$$

$$=-\int_0^{\frac{\pi}{2}}\log(\tan z)\,dz=-I_1\Rightarrow 2I_1=0\Rightarrow I_1=0$$

$$\Leftrightarrow I_2=\int_1^2\frac{\log x\,dx}{x}=\int_1^2\log x\,.\,d(\log x)=\left[\frac{\log^2 x}{2}\right]_1^2=\frac{\log^2 2}{2}$$

$$\Leftrightarrow I_3=\int_0^1\frac{x\tan^{-1}x}{(1+x^2)}dx\Rightarrow x=\tan z\Rightarrow dx=\sec^2 z\,dz$$

$$=\int_0^{\frac{\pi}{4}}z\tan z\,dz=[z\log(\sec z)]_0^{\frac{\pi}{4}}+\int_0^{\frac{\pi}{4}}\log(\cos z)\,dz$$

$$=\frac{\pi}{4}\log\left(\sec\frac{\pi}{4}\right)+\int_0^{\frac{\pi}{4}}\left[-\log 2-\sum_{n=1}^\infty\frac{(-1)^n\cdot\cos(2nz)}{n}\right]dz$$

$$=\frac{\pi}{8}\log 2-\frac{\pi}{4}\log 2-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}}\cos(2nz)\,dz$$

$$=-\frac{\pi}{8}\log 2-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n\cdot\sin\left(\frac{n\pi}{2}\right)}{n^2}=-\frac{\pi}{8}\log 2+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}=-\frac{\pi}{8}\log 2+\frac{G}{2}$$

$$\Leftrightarrow I_4=\int_0^1\frac{\log(1+x^2)}{(1+x^2)}dx=\int_0^{\frac{\pi}{4}}\log(1+\tan^2 z)\,dz=\int_0^{\frac{\pi}{4}}\log\left(\frac{\sin^2 z+\cos^2 z}{\cos^2 z}\right)dz$$

$$=-2\int_0^{\frac{\pi}{4}}\log(\cos z)\,dz=-2\left[-\frac{\pi}{4}\log 2+\frac{G}{2}\right]=\frac{\pi}{2}\log 2-G$$

$$\Leftrightarrow I_5=\int_0^{\infty}\frac{\log(x^2+1)}{(x^2+1)}dx=\int_0^{\frac{\pi}{4}}\log(1+\tan^2 z)\,dz=-2\int_0^{\frac{\pi}{4}}\log(\cos z)\,dz$$

$$=-2\left(-\frac{\pi}{2}\log 2\right)=\pi\log 2\Leftrightarrow I_6=\int_0^1\frac{\log^2(1+x^2)}{x}dx$$

$$=\left[\frac{\log(1+x^2)}{(-x)}\right]_0^1+\int_0^1\frac{2\log(1+x^2)\,2x}{(1+x^2)x}dx$$

$$=-\log^2 2+4\int_0^1\frac{\log(1+x^2)}{(1+x^2)}dx=-\log^2 2+4\left(\frac{\pi}{2}\log 2-G\right)$$

$$=-\log^2 2+2\pi\log 2-4G\Leftrightarrow-\frac{I_1}{4}-\frac{I_2}{2}-\frac{I_6}{4}+I_3+\frac{I_4}{2}+\frac{I_5}{8}$$

$$=-0-\frac{\log^2 2}{4}-\frac{1}{4}(\log^2 2+2\pi\log 2-4G)-\frac{\pi}{8}\log 2+\frac{G}{2}+\frac{1}{2}\left(\frac{\pi}{2}\log 2-G\right)+\frac{\pi}{8}\log 2$$

$$=-\frac{\pi}{2}\log 2+G-\frac{\pi}{8}\log 2+\frac{G}{2}+\frac{\pi}{4}\log 2-\frac{G}{2}+\frac{\pi}{8}\log 2=G-\frac{\pi}{4}\log 2$$

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704. Prove that:

$$\cos\left(\frac{2\pi}{13}\right)\cdot\cos\left(\frac{5\pi}{13}\right)\cdot\cos\left(\frac{6\pi}{13}\right)=\frac{\sqrt{13}-3}{16}$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Marian Ursărescu-Romania

Let ε *be the root of the equation* $x^{13}=1\Rightarrow$

$$\varepsilon=\cos\frac{2\pi}{13}+i\sin\frac{2\pi}{13}\Rightarrow\varepsilon^3=\cos\frac{6\pi}{13}+i\sin\frac{6\pi}{13}\text{ and}$$

$$\varepsilon^9=\cos\frac{18\pi}{13}+i\sin\frac{18\pi}{13}=\cos\left(\pi+\frac{5\pi}{13}\right)+i\sin\left(\pi+\frac{5\pi}{13}\right)=-\cos\frac{5\pi}{13}-i\sin\frac{5\pi}{13}\Rightarrow$$

$$-\varepsilon^9=\cos\frac{5\pi}{13}+i\sin\frac{5\pi}{13}\Rightarrow$$

$$\cos\frac{2\pi}{3}\cdot\cos\frac{5\pi}{13}\cdot\cos\frac{6\pi}{13}=-\frac{1}{8}\left(\varepsilon+\frac{1}{\varepsilon}\right)\left(\varepsilon^3+\frac{1}{\varepsilon}\right)\left(\varepsilon^9+\frac{1}{\varepsilon^9}\right)=$$

$$=-\frac{1}{8}(\varepsilon+\varepsilon^{12})(\varepsilon^3+\varepsilon^{10})(\varepsilon^9+\varepsilon^4)\Rightarrow\textit{we must show:}$$

$$-\frac{1}{8}(\varepsilon+\varepsilon^{12})(\varepsilon^3+\varepsilon^{10})(\varepsilon^9+\varepsilon^4)=\frac{\sqrt{13}-3}{16}\Leftrightarrow$$

$$(\varepsilon+\varepsilon^{12})(\varepsilon^3+\varepsilon^{10})(\varepsilon^9+\varepsilon^4)=\frac{3-\sqrt{13}}{2}\quad(1)$$

Let $z=(\varepsilon+\varepsilon^{12})(\varepsilon^3+\varepsilon^{10})(\varepsilon^9+\varepsilon^4)=(\varepsilon^4+\varepsilon^{11}+\varepsilon^3+\varepsilon^9)(\varepsilon^9+\varepsilon^4)$

$$=1+\varepsilon^8+\varepsilon^7+\varepsilon^2+\varepsilon^{11}+\varepsilon^6+\varepsilon^5+1\Rightarrow$$

$$z=\varepsilon^{11}+\varepsilon^8+\varepsilon^7+\varepsilon^6+\varepsilon^5+\varepsilon^2+2\quad(2)$$

$$z^2=\left(\varepsilon^{11}+\varepsilon^8+\varepsilon^7+\varepsilon^6+\varepsilon^5+\varepsilon^2+2\right)^2=$$

$$\varepsilon^9+\varepsilon^3+\varepsilon+\varepsilon^{12}+\varepsilon^{10}+\varepsilon^4+4+2\varepsilon^6+2\varepsilon^5+2\varepsilon^4+$$

$$+2\varepsilon^3+1+4\varepsilon^{11}+2\varepsilon^2+2\varepsilon+2+2\varepsilon^{10}+4\varepsilon^8+$$

$$+2+2\varepsilon^{12}+2\varepsilon^9+2\varepsilon^7+2\varepsilon^{11}+2\varepsilon^8+4\varepsilon^6+2\varepsilon^7+4\varepsilon^5+4\varepsilon^2=$$

$$=3\varepsilon^{12}+6\varepsilon^{11}+3\varepsilon^{10}+3\varepsilon^9+6\varepsilon^8+4\varepsilon^7+4\varepsilon^6+6\varepsilon^5+3\varepsilon^4+6\varepsilon^2+3\varepsilon+7\quad(2)$$

From (1)+(2)$\Rightarrow z^2=3z-1=0\Rightarrow\Delta=9+4=13\Rightarrow z_{1,2}=\frac{3\pm\sqrt{13}}{2}\Rightarrow z=\frac{3-\sqrt{13}}{2}$,

because $\frac{3+\sqrt{13}}{2}>1$

Solution 2 by Mokhtar Khassani-Mostaganem-Algerie

Let: $m = \cos\left(\frac{2\pi}{13}\right)\cos\left(\frac{5\pi}{13}\right)\cos\left(\frac{6\pi}{13}\right)$ *and:* $n = \cos\left(\frac{\pi}{13}\right)\cos\left(\frac{3\pi}{13}\right)\cos\left(\frac{9\pi}{13}\right)$

$$m + n = \cos\left(\frac{2\pi}{13}\right)\cos\left(\frac{5\pi}{13}\right)\cos\left(\frac{6\pi}{13}\right) + \cos\left(\frac{\pi}{13}\right)\cos\left(\frac{3\pi}{13}\right)\cos\left(\frac{\pi}{13}\right) =$$

$$= \frac{\cos\left(\frac{7\pi}{13}\right) + \cos\left(\frac{3\pi}{13}\right)}{2}\cos\left(\frac{6\pi}{13}\right) + \frac{\cos\left(\frac{4\pi}{13}\right) + \cos\left(\frac{2\pi}{13}\right)}{2}\cos\left(\frac{9\pi}{13}\right)$$

$$\Rightarrow 4(m+n) = -1 + \cos\left(\frac{\pi}{13}\right) + \cos\left(\frac{3\pi}{13}\right) + \cos\left(\frac{2\pi}{13}\right) + \cos\left(\frac{5\pi}{13}\right) +$$

$$+ \cos\left(\frac{7\pi}{13}\right) + \cos\left(\frac{9\pi}{13}\right) + \cos\left(\frac{11\pi}{13}\right) = -1 - \frac{1}{2} \Rightarrow 8(m+n) = -3$$

$\therefore$ *Note:* $\cos\left(\frac{9\pi}{13}\right) = -\cos\left(\frac{4\pi}{13}\right), \cos\left(\frac{5\pi}{13}\right) = -\cos\left(\frac{8\pi}{13}\right)$ *and* $\cos\left(\frac{3\pi}{13}\right) = -\cos\left(\frac{16\pi}{13}\right)$ *also:*

$$\sum_{n=0}^{k}\cos((2n+1)z) = \frac{1}{2}\csc z \sin(2(k+1)z)$$

$$\Rightarrow 2mn\sin\left(\frac{\pi}{13}\right) = -2\sin\left(\frac{\pi}{13}\right)\cos\left(\frac{\pi}{13}\right)\cos\left(\frac{2\pi}{13}\right)\cos\left(\frac{4\pi}{13}\right)\cos\left(\frac{6\pi}{13}\right)\cos\left(\frac{8\pi}{13}\right)\cos\left(\frac{16\pi}{13}\right) =$$

$$= -\frac{1}{16}\sin\left(\frac{32\pi}{13}\right)\cos\left(\frac{6\pi}{13}\right) = \frac{1}{16}\sin\left(\frac{7\pi}{13}\right)\cos\left(\frac{7\pi}{13}\right) = -\frac{1}{32}\sin\left(\frac{\pi}{13}\right)$$

$\begin{cases} 8(m+n) = -3 \\ 64mn = -1 \end{cases}$ m *and* n *are the roots of the equation:* $64x^2 + 24x - 1 = 0$,

$$m > 0 \Rightarrow m = \frac{\sqrt{13} - 3}{16}$$

Something more generic: $\prod_v^{u-1}\cos\left(\frac{2v+1}{2u+1}\pi\right) = \frac{(-1)^p}{2^u}$ *where:* $p = \left[\frac{u}{2}\right], [\,.\,]$ *GIF*

in my solution we can take: $u = 6$ *we find:* $64mn = -1$

705. Prove that:

$$5\Psi_1\left(\frac{1}{3}\right) + \Psi_1\left(\frac{5}{6}\right) = \frac{16}{3}\pi^2$$

where $\Psi_1(x)$ is the trigamma function.

Proposed by Vasile Mircea Popa – Romania

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Solution 1 by Abdul Hafeez Ayinde-Nigeria

From ERF, we obtain $\psi_1(1-z)+\psi_1(z)=\frac{\pi^2}{\sin^2(\pi z)} \Rightarrow \psi_1\left(\frac{1}{6}\right)+\psi_1\left(\frac{5}{6}\right)=\frac{\pi^2}{\sin^2\left(\frac{\pi}{6}\right)}=4\pi^2$

$$\psi_1\left(\frac{5}{6}\right)=4\pi^2-\psi_1\left(\frac{1}{6}\right)$$

$\Gamma(3z)=\frac{3^{3z-\frac{1}{2}}}{2\pi}\Gamma(z)\Gamma\left(z+\frac{1}{3}\right)\Gamma\left(z+\frac{2}{3}\right)$ *is a duplication formula.*

$$9\psi_1(3z)=\psi_1(z)+\psi_1\left(z+\frac{1}{3}\right)+\psi_1\left(z+\frac{2}{3}\right)$$

$$9\psi_1(1)=\psi_1\left(\frac{1}{3}\right)+\psi_1\left(\frac{2}{3}\right)+\psi_1(1)$$

$$8\psi_1(1)=\psi_1\left(\frac{1}{3}\right)+\psi_1\left(\frac{2}{3}\right)$$

$$\frac{4\pi^2}{3}=\psi_1\left(\frac{1}{3}\right)+\psi_1\left(\frac{2}{3}\right)$$

$\Gamma\left(z+\frac{1}{2}\right)=\frac{\Gamma(2z)\sqrt{\pi}}{\Gamma(z)2^{2z-1}}$ *is also a duplication formula.*

$$\psi_1\left(z+\frac{1}{2}\right)=4\psi_1(2z)-\psi_1(z)$$

$$\psi_1\left(\frac{2}{3}\right)=4\psi_1\left(\frac{1}{2}\right)-\psi_1\left(\frac{1}{6}\right)$$

$$\psi_1\left(\frac{1}{6}\right)=4\psi_1\left(\frac{1}{3}\right)-\psi_1\left(\frac{2}{3}\right)$$

$$\frac{1}{4}\psi_1\left(\frac{1}{6}\right)+\frac{1}{4}\psi_1\left(\frac{2}{3}\right)=\psi_1\left(\frac{1}{3}\right)$$

$$\frac{5}{4}\psi_1\left(\frac{1}{6}\right)+\frac{5}{4}\psi_1\left(\frac{2}{3}\right)=5\psi_1\left(\frac{1}{3}\right)$$

We have proven $\psi_1\left(\frac{5}{6}\right)=4\pi^2-\psi_1\left(\frac{1}{6}\right)$

$$5\psi_1\left(\frac{1}{3}\right)+\psi_1\left(\frac{5}{6}\right)=4\pi^2+\frac{1}{4}\psi_1\left(\frac{1}{6}\right)+\frac{5}{4}\psi_1\left(\frac{2}{3}\right)$$

$$5\psi_1\left(\frac{1}{3}\right)+\psi_1\left(\frac{5}{6}\right)=4\pi^2+\frac{1}{4}\left(4\psi_1\left(\frac{1}{3}\right)-\psi_1\left(\frac{2}{3}\right)\right)+\frac{5}{4}\psi_1\left(\frac{2}{3}\right)$$

$$5\psi_1\left(\frac{1}{3}\right)+\psi_1\left(\frac{5}{6}\right)=4\pi^2+\psi_1\left(\frac{1}{3}\right)-\frac{1}{4}\psi_1\left(\frac{2}{3}\right)+\frac{5}{4}\psi_1\left(\frac{2}{3}\right)$$

$$5\psi_1\left(\frac{1}{3}\right)+\psi_1\left(\frac{5}{6}\right)=4\pi^2+\psi_1\left(\frac{1}{3}\right)+\psi_1\left(\frac{2}{3}\right)$$

$$5\psi_1\left(\frac{1}{3}\right)+\psi_1\left(\frac{5}{6}\right)=4\pi^2+\frac{4\pi^2}{3}=\frac{16\pi^2}{3}$$

Solution 2 by Mokhtar Khassani-Mostaganem-Algerie

$$5\Psi_1\left(\frac{1}{3}\right)+\Psi_1\left(\frac{5}{6}\right)=4\Psi_1\left(\frac{1}{3}\right)+\Psi_1\left(\frac{1}{3}\right)+\Psi_1\left(\frac{1}{3}+\frac{1}{2}\right)=4\left(\Psi_1\left(\frac{1}{3}\right)+\Psi_1\left(1-\frac{1}{3}\right)\right)=$$

$$=4\frac{\pi^2}{\sin^2\left(\frac{\pi}{3}\right)}=\frac{16}{3}\pi^2$$

Note: $\Psi_1(z)+\Psi_1\left(z+\frac{1}{2}\right)=4\Psi_1(2z)$ ***and*** $\Psi_1(z)+\Psi_1(1-z)=\frac{\pi^2}{\sin^2(\pi z)}$

Solution 3 by Nelson Javier Villaherrera Lopez-El Salvador

$$5\psi_1\left(\frac{1}{3}\right)+\psi_1\left(\frac{5}{6}\right)=5\psi_1\left(\frac{1}{3}\right)+\psi_1\left(\frac{1}{3}+\frac{1}{2}\right)=5\psi_1\left(\frac{1}{3}\right)+\left\{\ln\left[\Gamma\left(x+\frac{1}{2}\right)\right]\right\}''\Big|_{x=\frac{1}{3}}=$$

$$=5\psi_1\left(\frac{1}{3}\right)+\left\{\ln\left[\frac{2^{1-2x}\sqrt{\pi}\Gamma(2x)}{\Gamma(x)}\right]\right\}''\Bigg|_{x=\frac{1}{3}}$$

$$=5\psi\left(\frac{1}{3}\right)+\left\{(1-2x)\ln(2)+\frac{\ln(\pi)}{2}+\ln[\Gamma(2z)]-\ln[\Gamma(x)]\right\}''\Bigg|_{x=\frac{1}{3}}=$$

$$=5\psi_1\left(\frac{1}{3}\right)+[-2\ln(2)+2\psi_0(2x)-\psi_0(x)]'|_{x=\frac{1}{3}}$$

$$=5\psi_1\left(\frac{1}{3}\right)+[4\psi_1(2x)-\psi(x)]'|_{x=\frac{1}{3}}=5\psi\left(\frac{1}{3}\right)+4\psi_1\left(\frac{2}{3}\right)-\psi_1\left(\frac{1}{3}\right)=$$

$$=4\left[\psi_1\left(\frac{1}{3}\right)+\psi_1\left(\frac{2}{3}\right)\right]=4\{\ln[\Gamma(x)\Gamma(1-x)]\}''\,|_{x=\frac{1}{3}}$$

$$=4\left\{\ln\left[\frac{\pi}{\sin(\pi x)}\right]\right\}''\Bigg|_{x=\frac{1}{3}}=4\{\ln(\pi)-\ln[\sin(\pi x)]\}''|_{x=\frac{1}{3}}=$$

$$=4\pi\left[-\frac{\cos(\pi x)}{\sin(\pi x)}\right]'\Bigg|_{x=\frac{1}{3}}=-4\pi\left[\frac{-\pi\sin^2(\pi x)-\pi\cos^2(\pi x)}{\sin^2(\pi x)}\right]\Bigg|_{x=\frac{1}{3}}$$

$$=4\pi^2\left[\frac{1}{\sin^2(\pi x)}\right]\Bigg|_{x=\frac{1}{3}}=\frac{4\pi^2}{\sin^2\left(\frac{\pi}{3}\right)}=\frac{4\pi^2}{\frac{3}{4}}=\frac{16}{3}\pi^2$$

706. Prove that:

$$_2F_1(\alpha,\beta,\beta-\alpha+1,-1)=\frac{\Gamma(\beta-\alpha+1)\Gamma\left(\frac{\beta}{2}+1\right)}{\Gamma(\beta+1)\Gamma\left(\frac{\beta}{2}-\alpha+1\right)}$$

Proposed by Nawar Alasadi-Babylon-Iraq

Solution by proposer

$$\because\ _2F_1(\alpha,\beta,\gamma,x)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{n!\,(\gamma)_n}x^n$$

Where $(\alpha)_n=\alpha(\alpha+1)\dots(\alpha+n-1)=\frac{1\cdot2\cdot\ldots\cdot(\alpha-1)\alpha(\alpha+1)(\alpha+n-1)}{1\cdot2\cdot\ldots\cdot(\alpha-1)}=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$

$$\therefore\frac{(\beta)_n}{(\gamma)_n}=\frac{\Gamma(\beta+n)}{\Gamma(\beta)}\cdot\frac{\Gamma(\gamma)}{\Gamma(\gamma+n)}=\frac{\Gamma(\gamma)\Gamma(\beta+n)\Gamma(\gamma-\beta)}{\Gamma(\beta)\Gamma(\gamma+n)\Gamma(\gamma-\beta)}=$$

$$=\frac{\Gamma(\gamma)}{\Gamma(\beta)}\frac{\Gamma(\beta+n)\Gamma(\gamma-\beta)}{\Gamma(\beta+n+\gamma-\beta)}\cdot\frac{1}{\Gamma(\gamma-\beta)}$$

$$\frac{(\beta)_n}{(\gamma)_n}=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1(1-t)^{\gamma-\beta-1}t^{\beta+n-1}\,dt=$$

$$=\frac{1}{\frac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\beta+\gamma-\beta)}}\int_0^1 t^{\beta+n-1}(1-t)^{\gamma-\beta-1}dt=\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta+n-1}(1-t)^{\gamma-\beta-1}dt$$

$$\therefore\ _2F_1(\alpha,\beta,\gamma,x)=\sum_{n=0}^{\infty}\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta+n-1}\int_0^1 t^{\beta+n-1}(1-t)^{\gamma-\beta-1}\left(\frac{(\alpha)_n x^n}{n!}\right)dt=$$

$$=\sum_{n=1}^{\infty}\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}\cdot\frac{(\alpha)_n(xt)^n}{n!}dt$$

$$_2F_1=\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}\sum_{n=0}^{\infty}\frac{(\alpha)_n(xt)^n}{n!}dt=$$

$$=\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}\left\{1+axt+\frac{\alpha(\alpha+1)}{2!}(xt)^2+\cdots\right\}dt$$

$$ {}_2F_1(\alpha,\beta,\gamma,x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}dt, |x| < 1, \gamma > \beta > 0 $$

Putting $x = -1$ ***and*** $\gamma = \beta - \alpha + 1$

$$ {}_2F_1(\alpha,\beta,\beta-\alpha+1,-1) = \frac{1}{\beta(\beta,1-\alpha)}\int_0^1 t^{\beta-1}(1-t)^{-\alpha}(1+t)^{-\alpha}dt = $$

$$ = \frac{1}{\frac{\Gamma(\beta)\Gamma(1-\alpha)}{\Gamma(\beta-\alpha+1)}}\int_0^1 t^{\beta-1}(1-t^2)^{-\alpha}dt $$

Let $z = t^2 \Rightarrow dz = 2t\,dt$

$$ {}_2F_1(\alpha,\beta,\beta-\alpha+1,-1) = \frac{1}{2}\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)\Gamma(1-\alpha)}\int_0^1 z^{\frac{\beta}{2}-1}(1-z)^{(1-\alpha)-1}dz = $$

$$ = \frac{1}{2}\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)\Gamma(1-\alpha)}\cdot B\left(\frac{\beta}{2},1-\alpha\right) $$

$$ {}_2F_1(\alpha,\beta,\beta-\alpha+1,-1) = \frac{1}{2}\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)\Gamma(1-\alpha)}\cdot\frac{\Gamma\left(\frac{\beta}{2}\right)\Gamma(1-\alpha)}{\Gamma\left(\frac{\beta}{2}-\alpha+1\right)} = \frac{\Gamma(\beta-\alpha+1)\cdot\frac{\beta}{2}\cdot\Gamma\left(\frac{\beta}{2}\right)}{\alpha\Gamma(\beta)\Gamma\left(\frac{\beta}{2}-\alpha+1\right)} $$

$$ \therefore {}_2F_1(\alpha,\beta,\beta-\alpha+1,-1) = \frac{\Gamma(\beta-\alpha+1)\Gamma\left(\frac{\beta}{2}+1\right)}{\Gamma(\beta+1)\Gamma\left(\frac{\beta}{2}-\alpha+1\right)} $$

707. Prove that:

$$ \sqrt[3]{1+\cos\frac{4\pi}{19}+\cos\frac{6\pi}{19}-\cos\frac{9\pi}{19}} + \sqrt[3]{1+\cos\frac{2\pi}{19}-\cos\frac{3\pi}{19}-\cos\frac{5\pi}{19}} + $$

$$ + \sqrt[3]{1-\cos\frac{\pi}{19}-\cos\frac{7\pi}{19}+\cos\frac{8\pi}{19}} = \frac{1}{2}\sqrt[3]{12\sqrt[3]{19}-4} $$

Proposed by Vasile Mircea Popa-Romania

Solution by Ruangkhaw Chaoka-Chiangrai-Thailand

$$\textit{Or } \sqrt[3]{2-2\left(\cos\frac{9\pi}{19}+\cos\frac{13\pi}{19}+\cos\frac{15\pi}{19}\right)}+\sqrt[3]{2-2\left(\cos\frac{3\pi}{19}+\cos\frac{5\pi}{19}+\cos\frac{17\pi}{19}\right)}$$

$$+\sqrt[3]{2-2\left(\cos\frac{\pi}{19}+\cos\frac{7\pi}{19}+\cos\frac{11\pi}{19}\right)}\overset{??}{=}\sqrt[3]{3\sqrt[3]{19}-1}\ \ (a)$$

$$\because\begin{cases} x_1=2\left(\cos\frac{9\pi}{19}+\cos\frac{13\pi}{19}+\cos\frac{15\pi}{19}\right)\\ x_2=2\left(\cos\frac{3\pi}{19}+\cos\frac{5\pi}{19}+\cos\frac{17\pi}{19}\right)\\ x_3=2\left(\cos\frac{\pi}{19}+\cos\frac{7\pi}{19}+\cos\frac{11\pi}{19}\right)\end{cases}\textit{ be the roots of } x^3-x^2-6x+7=0\ \ (b)$$

Let $y_1=2-x_1, y_2=2-x_2$ *and* $y_3=2-x_3\Rightarrow$ *(a)* $\sqrt[3]{y_1}+\sqrt[3]{y_2}+\sqrt[3]{y_3}\overset{??}{=}\sqrt[3]{3\cdot\sqrt[3]{19}-1}$

$y\to 2-x\Rightarrow$ *(b);* $y^3-5y^2+2y+1=0$ *(c) and let* $a=\sqrt[3]{y_1}, b=\sqrt[3]{y_2}$ *and* $c=\sqrt[3]{y_3}$

$$\textit{Let } p,q,r\in\mathbb{R}\begin{cases}p=a+b+c\\ q=ab+bc+ca\\ r=abc\end{cases}\Leftrightarrow (c);\begin{cases}a^3+b^3+c^3=5\\ a^3b^3+b^3c^3+c^3a^3=2\\ a^3b^3c^3=-1\Rightarrow r=-1\end{cases}$$

$$\because\begin{cases}a^3+b^3+c^3=p^3-3pq+3r=5\Rightarrow p^3-3pq=8\to(1)\\ a^3b^3+b^3c^3+c^3a^3=q^3-3pqr+3r^2=2\Rightarrow q^3+3pq=-1\to(2)\end{cases}$$

$$(1)+(2): p^3+q^3=7\Leftrightarrow p^6+p^3q^3=7p^3\Leftrightarrow 27p^6+27p^3q^3=189p^3\Leftrightarrow$$

$$\Leftrightarrow 27p^6+(p^3-8)^3=189p^3$$

$$p^9+3p^6+3p^3-512=0\Leftrightarrow p^9+3p^6+3p^3+1=513\Leftrightarrow (p^3+1)^3=513\Leftrightarrow$$

$$\Leftrightarrow p^3+1=3\cdot\sqrt[3]{19}\ \therefore\ p=\sqrt[3]{3\cdot\sqrt[3]{19}-1}$$

708. Prove that:

$$\sin\frac{2\pi}{13}-\sin\frac{5\pi}{13}+\sin\frac{6\pi}{13}=\frac{1}{2}\sqrt{\frac{13-3\sqrt{13}}{2}}$$

Proposed by Vasile Mircea Popa – Romania

Solution by Marian Ursărescu – Romania

Let ε *be the root of equation* $x^{13}=1\Rightarrow$

$$\varepsilon=\cos\frac{2\pi}{13}+i\sin\frac{2\pi}{13}\Rightarrow\varepsilon^{-1}=\frac{1}{\varepsilon}=\cos\frac{2\pi}{13}-i\sin\frac{2\pi}{13}\Rightarrow$$

$$\varepsilon-\frac{1}{\varepsilon}=2i\sin\frac{2\pi}{13}\Rightarrow\sin\frac{2\pi}{13}=\frac{1}{2i}\left(\varepsilon-\frac{1}{\varepsilon}\right)\ (1)$$

Now: $\varepsilon^9=\cos\frac{18\pi}{13}+i\sin\frac{18\pi}{13}=\cos\left(\pi+\frac{5\pi}{13}\right)+i\sin\left(\pi+\frac{5\pi}{13}\right)$

$$=-\cos\frac{5\pi}{13}-i\sin\frac{5\pi}{13}\Rightarrow-\varepsilon^9=\cos\frac{5\pi}{13}+i\sin\frac{5\pi}{13}\Rightarrow$$

$$-\varepsilon^9+\frac{1}{\varepsilon^9}=2i\sin\frac{5\pi}{13}\Rightarrow\sin\frac{5\pi}{13}=-\frac{1}{2i}\left(\varepsilon^9-\frac{1}{\varepsilon^9}\right)\ (2)$$

$$\varepsilon^3=\cos\frac{6\pi}{13}+i\sin\frac{6\pi}{13}\Rightarrow\varepsilon^{-3}=\frac{1}{\varepsilon^3}=\cos\frac{6\pi}{13}-i\sin\frac{6\pi}{13}\Rightarrow$$

$$\varepsilon^3-\frac{1}{\varepsilon^3}=2i\sin\frac{6\pi}{13}\Rightarrow\sin\frac{6\pi}{13}=\frac{1}{2i}\left(\varepsilon^3-\frac{1}{\varepsilon^3}\right)\ (3)$$

From (1)+(2)+(3) $\Rightarrow\sin\frac{2\pi}{13}-\sin\frac{5\pi}{13}+\sin\frac{6\pi}{13}=$

$$=\frac{1}{2i}\left(\varepsilon-\frac{1}{\varepsilon}+\varepsilon^9-\frac{1}{\varepsilon^9}+\varepsilon^3-\frac{1}{\varepsilon^3}\right)=\frac{1}{2i}(\varepsilon-\varepsilon^{12}+\varepsilon^9-\varepsilon^4+\varepsilon^3-\varepsilon^{10})\ (4)$$

Let $z=\varepsilon-\varepsilon^{12}+\varepsilon^9-\varepsilon^4+\varepsilon^3-\varepsilon^{10}\Rightarrow z^2=(\varepsilon-\varepsilon^{12}+\varepsilon^9-\varepsilon^4+\varepsilon^3-\varepsilon^{10})^2=$

$$2\varepsilon^{12}-\varepsilon^{11}+2\varepsilon^{10}+2\varepsilon^9-\varepsilon^8-\varepsilon^7-\varepsilon^6-\varepsilon^5+2\varepsilon^4+2\varepsilon^3-\varepsilon^2+2\varepsilon-6\ (5)$$

$$\Rightarrow z^4=(z^2)^2=-25\varepsilon^{12}+14\varepsilon^{11}-25\varepsilon^{10}-25\varepsilon^9+14\varepsilon^8+14\varepsilon^7+$$

$$+14\varepsilon^6+14\varepsilon^5-24\varepsilon^4-25\varepsilon^3+14\varepsilon^2-25\varepsilon+66\ (6)$$

From (5)+(6)$\Rightarrow z^4+13z^2+13=0;z^2=t\Rightarrow$

$$z=i\sqrt{\frac{13-3\sqrt{13}}{2}}\ (7)$$

From (4)+(7)$\Rightarrow\sin\frac{2\pi}{13}-\sin\frac{5\pi}{13}+\sin\frac{6\pi}{13}=\frac{1}{2}\sqrt{\frac{13-3\sqrt{13}}{2}}$

709. Inspired by Srinivasa Raghava

Let $T(n)$ be the n^th triangular number and $M=2\sum_{n=1}^{\infty}\frac{T(n)}{T(T(T(n)))}\left(n+\frac{1}{2}\right)$

Prove that:

$$\prod_{k=2}^{\infty}\left(1-\frac{1}{k^M}\right)=\frac{1}{4\pi}\sum_{r=0}^{\infty}\frac{\pi^{2r+1}}{(2r+1)!}=\frac{\sinh(\pi)}{4\pi}$$

Proposed by Naren Bhandari-Bajura-Nepal

Solution by proposer

As $T(n)$ is nth triangular number, so, we have: $2T(n) = n^2 + n$ and we shall show that: $\frac{T(n)}{T(T(T(n)))} = \frac{64}{(n^2+n+2)(n^4+2n^3+3n^2+2n+8)}$. *Since* $2T(n) = n^2 + n$, *so:*

$T(T(n)) = \frac{T(n)}{2}(T(n) + 1) = \frac{1}{8}\left((n^2 + n)(n^2 + n + 2)\right)$ *and hence*

$$T\left(T(T(n))\right) = \frac{T(T(n))}{2}\left(T(T(n)) + 2\right)$$

$$= \frac{1}{128}\left((n^2 + n)(n^2 + n + 2)\{(n^2 + n)(n^2 + n + 2) + 8\}\right)$$

therefore, we have: $\frac{T(n)}{T(T(T(n)))} = \frac{64}{(n^2+n+2)\left((n^2+n)(n^2+n+2)+8\right)} = \frac{64}{(n^2+n+2)(n^4+2n^3+3n^2+2n+8)}$

Further

$$2\sum_{n=1}^{\infty}\frac{T(n)}{T\left(T(T(n))\right)}\left(n + \frac{1}{2}\right) = \sum_{n=1}^{\infty}\frac{64(2n + 1)}{(n^2 + n + 2)(n^4 + 2n^3 + 3n^2 + 2n + 8)}$$

Since $n^4 + 2n^3 + 3n^2 + 2n + 8 = (n^2 - n + 2)(n^2 + 3n + 4)$. *Therefore,*

$$M = \sum_{n=1}^{\infty}\frac{64(2n + 1)}{(n^2 + n + 2)(n^2 - n + 2)(n^2 + 3n + 4)}$$

After partial fraction decomposition of the summand we have:

$$M = 8\sum_{n=1}^{\infty}\left(\frac{2n + 1}{n^2 + n + 2} - \frac{n - 1}{n^2 - n + 2} + \frac{n + 1}{n^2 + 3n + 4}\right)$$

$= 8\left(\frac{3}{4} - \frac{0}{2} - \frac{2}{8} + \sum_{n=2}^{\infty}\left(\frac{2n+1}{n^2+n+2} - \frac{n}{n^2+n+2} - \frac{n+1}{n^2+n+2}\right)\right) = 4$. *So, we will evaluate:*

$$\prod_{k=2}^{\infty}\left(1 - \frac{1}{n^4}\right) = \prod_{k=2}^{\infty}\left(1 + \frac{1}{k^2}\right)\prod_{k=2}^{\infty}\left(1 - \frac{1}{k^2}\right)$$

The latter product is easy to evaluate as:

$$\lim_{m\to\infty}\prod_{k=2}^{\infty}\left(1 - \frac{1}{k^2}\right) = \lim_{m\to\infty}\left(\frac{1}{2} + \frac{1}{2m}\right) = \frac{1}{2}$$

And to evaluate the former product we shall be using Euler Product formula which is as $\frac{\sin x}{x} = \prod_{k=1}^{\infty}\left(1 - \frac{x^2}{k^2\pi^2}\right)$. *Setting* $x = i\pi$ *where* $i = \sqrt{-1}$ *we have:*

$$\frac{\mathfrak{I}(e^{ix})}{i\pi}=\frac{e^{-\pi}-e^{\pi}}{-2\pi}=2\prod_{k=2}^{\infty}\left(1+\frac{1}{k^2}\right)$$

which is

$$\prod_{k=2}^{\infty}\left(1+\frac{1}{k^2}\right)=\frac{e^{\pi}-e^{-\pi}}{2\pi}=\frac{1}{2\pi}\sum_{r=0}^{\infty}\frac{\pi^{2r+1}}{(2r+1)!}=\frac{\sinh(\pi)}{2\pi}$$

Finally, we get:

$$\prod_{k=2}^{\infty}\left(1-\frac{1}{n^4}\right)=\frac{1}{4\pi}\sum_{r=0}^{\infty}\frac{\pi^{2r+1}}{(2r+1)!}=\frac{\sinh(\pi)}{4\pi}$$

710. **Solve for $x\in\left(0,\frac{\pi}{2}\right)$:**

$$\sin x+\cos x+\tan x+\cot x+\frac{1}{2}(\sec x+\csc x)=2\left(1+\sqrt{2}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Khaled Abd Imouti-Damascus-Syria

$$\sin x+\cos x+\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}+\frac{1}{2}\left(\frac{1}{\cos x}+\frac{1}{\sin x}\right)=2\left(1+\sqrt{2}\right)$$

$$\sin x+\cos x+\frac{\sin^2 x+\cos^2 x}{\sin x\cdot\cos x}+\frac{1}{2}\cdot\frac{\sin x+\cos x}{\sin x\cdot\cos x}=2\left(1+\sqrt{2}\right)$$

$$\sin x+\cos x+\frac{1}{\sin x\cdot\cos x}+\frac{1}{2}\cdot\frac{\sin x+\cos x}{\sin x\cdot\cos x}=2\left(1+\sqrt{2}\right)$$

$$\sin x+\cos x+\frac{2+\sin x\cdot\cos x}{2\sin x\cdot\cos x}=2\left(1+\sqrt{2}\right)$$

Suppose: $y=\sin x+\cos x$

$$y^2=1+2\sin x\cdot\cos x\Rightarrow\frac{y^2-1}{2}=\sin x\cdot\cos x$$

$$\frac{y}{1}+\frac{2+y}{y^2-1}=2\left(1+\sqrt{2}\right)$$

$$\frac{y^3-y+2+y}{y^2-1}=2\left(1+\sqrt{2}\right)\Rightarrow\frac{y^3+2}{y^2-1}=2+2\sqrt{2}$$

$$y^3+2=\left(2+2\sqrt{2}\right)y^2-\left(2+2\sqrt{2}\right)$$

$$y^3+2=(2+2\sqrt{2})y^2-2-2\sqrt{2}$$

$$y^3-(2+2\sqrt{2})y^2+4+2\sqrt{2}=0 \quad (*)$$

Note: $y=\sqrt{2}$ ***satisfying the equation (*):***

$$2\sqrt{2}-2(2+2\sqrt{2})+4+2\sqrt{2}=2\sqrt{2}-4-4\sqrt{2}+4+2\sqrt{2}=0$$

$$y^2+(-2-2\sqrt{2})y$$

$$-----------$$

$$y-\sqrt{2} \quad | \, y^3-(2+2\sqrt{2})y^2+4+2\sqrt{2}$$

$$----$$

$$\mp y^3\pm\sqrt{2}\cdot y^2$$

$$-------$$

$$(-2-\sqrt{2})y^2+4+2\sqrt{2}$$

$$\mp(-2-\sqrt{2})y^2\mp 4\mp 2\sqrt{2}$$

$$-----------$$

$$0$$

$$(y-\sqrt{2})(y^2-(2+\sqrt{2})y)=0$$

$$y\cdot\left(y-(2+\sqrt{2})\right)\cdot(y-\sqrt{2})=0\to y=0\Rightarrow \sin x+\cos x=0$$

$$\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)=0$$

$x-\frac{\pi}{4}=\frac{\pi}{2}+\pi k\Rightarrow x=\frac{3\pi}{4}+\pi k$ ***impossible*** $\to y=2+\sqrt{2}$

$$\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)=\sqrt{2}(1+\sqrt{2})$$

$\cos\left(x-\frac{\pi}{4}\right)=1+\sqrt{2}>0$ ***impossible*** $\to y=\sqrt{2}$

$$\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)=\sqrt{2}$$

$$\cos\left(x-\frac{\pi}{4}\right)=1\Rightarrow x-\frac{\pi}{4}=2\pi k$$

$$x=\frac{\pi}{4}+2\pi k$$

For $k=0\Rightarrow x=\frac{\pi}{4}\in\left]0,\frac{\pi}{2}\right[$

So: the equation has a unique solution in the $\left]0,\frac{\pi}{2}\right[$

$$S=\left\{\frac{\pi}{4}\right\}$$

Solution 2 by Sohini Mondal-India

$$\tan x+\cot x=\left(\sqrt{\tan x}+\sqrt{\cot x}\right)^2-2$$

$$\because\left(\sqrt{\tan x}+\sqrt{\cot x}\right)\geq 0$$

$$\therefore \tan x+\cot x\geq 2$$

Again, $\frac{\sin x+\cos x}{2}\geq\sqrt{\sin x\cdot\cos x}$ *[as* $AM\geq GM$*]*

$$\Rightarrow(\sin x+\cos x)\left(1+\frac{1}{2\sin x\cdot\cos x}\right)\geq 2\sqrt{\sin x\cdot\cos x}\left(1+\frac{1}{2\sin x\cdot\cos x}\right)$$

$$\Rightarrow f(x)=\sin x+\cos x+\frac{1}{2}(\sec x+\csc x)\geq\left(2\sqrt{\sin x\cdot\cos x}+\frac{1}{\sqrt{\sin x\cdot\cos x}}\right)$$

Let, $\sqrt{\sin x\cdot\cos x}=t$

$\therefore f(t)=2t+\frac{1}{t};\ f'(t)=2-\frac{1}{t^2};\ f''(t)=\frac{2}{t^3}$. *If* $f'(t)=0$, *then* $t=\frac{1}{\sqrt{2}}$

Now, $f''(t)=2\sqrt{2}\therefore f''(t)>0$

$$\Leftrightarrow\sqrt{\sin x\cdot\cos x}=\frac{1}{\sqrt{2}}\Rightarrow\sin x\cdot\cos x=\frac{1}{2}$$

$$\Rightarrow 2\sin x\cdot\cos x=1\Rightarrow\sin 2x=1\Rightarrow x=\frac{\pi}{4}\left[x\in\left(0,\frac{\pi}{2}\right)\right]$$

$$f(x)=\sin x+\cos x+\frac{1}{2}(\sec x+\csc x)\geq\left(2\sqrt{\sin\frac{\pi}{4}\cdot\cos\frac{\pi}{4}}+\frac{1}{\sqrt{\sin\frac{\pi}{4}\cdot\cos\frac{\pi}{4}}}\right)=2\sqrt{2}$$

$$\therefore f(x)\geq 2\sqrt{2}$$

$\tan x+\cot x\geq 2$ *equality holds for* $x=\frac{\pi}{4}$ *as* $x\in\left(0,\frac{\pi}{2}\right)$

$\sin x+\cos x+\frac{1}{2}(\sec x+\csc x)\geq 2\sqrt{2}$ *minima found at* $x=\frac{\pi}{4}$ *where minimum value of the expression is* $2\sqrt{2}$*, so equality holds for* $x=\frac{\pi}{4}$ *as* $x\in\left(0,\frac{\pi}{2}\right)$

So, in, $\sin x+\cos x+\tan x+\cot x+\frac{1}{2}(\sec x+\csc x)\geq 2\left(1+\sqrt{2}\right)$ *equality holds for* $x=\frac{\pi}{4}$. *So, only solution* $x=\frac{\pi}{4}$

Solution 3 by Tran Hong-Dong Thap-Vietnam

$$x \in \left(0; \frac{\pi}{2}\right) \Rightarrow \sin x ; \cos x > 0$$

$$\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \overset{(AM-GM)}{\geq} 2\sqrt{\frac{\sin x}{\cos x} \cdot \frac{\cos x}{\sin x}} = 2$$

$$\sin x + \cos x + \frac{1}{2}(\sec x + \csc x) = \sin x + \cos x + \frac{1}{2}\left(\frac{1}{\cos x} + \frac{1}{\sin x}\right)$$

$$= (\sin x + \cos x)\left(1 + \frac{1}{2 \cdot \cos x \cdot \sin x}\right)$$

$$\overset{(AM-GM)}{\geq} 2\sqrt{\sin x \cdot \cos x}\left(1 + \frac{1}{2 \cdot \cos x \cdot \sin x}\right) \overset{(t=\sqrt{\sin x \cdot \cos x})}{=} 2t + \frac{1}{t} \overset{(AM-GM)}{\geq} 2\sqrt{2}$$

$$\Rightarrow \sin x + \cos x + \tan x + \cot x + \frac{1}{2}(\sec x + \csc x) \geq 2\sqrt{2}$$

$$\textit{Equality} \Leftrightarrow \begin{cases} \tan x = \cot x \\ 2\sqrt{\sin x \cdot \cos x} = \frac{1}{\sqrt{\sin x \cdot \cos x}} \end{cases} \overset{0<x<\frac{\pi}{2}}{\Leftrightarrow} x = \frac{\pi}{4}$$

Solution 4 by Sudhir Jha-Kolkata-India

$$\tan x + \cot x \geq 2\sqrt{\tan x \cot x} = 2 \quad \textit{(I)}$$

Equality holds for $x = \frac{\pi}{4}$

$$\sin x + \cos x \geq 2\sqrt{\sin x \cos x} = \sqrt{2 \sin x} \quad \textit{(II)}$$

$$\frac{\sec x + \csc x}{2} \geq \sqrt{\sec x \csc x} = \sqrt{\frac{2}{2 \sin x \cos x}} \Rightarrow \frac{1}{2}(\sec x + \csc x) \geq \sqrt{\frac{2}{\sin 2x}} \quad \textit{(III)}$$

$$\textit{(II)}+\textit{(III)} \Rightarrow \sin x + \cos x + \frac{1}{2}(\sec x + \csc x) \geq \sqrt{2}\left(\sqrt{\sin 2x} + \frac{1}{\sqrt{\sin 2x}}\right)$$

$$\Rightarrow \sin x + \cos x + \frac{1}{2}(\sec + \csc x) \geq \sqrt{2}(2) \quad \textit{(IV)}$$

$$\left(\because \sqrt{\sin 2x} + \frac{1}{\sqrt{\sin 2x}} \geq 2\sqrt{\sqrt{\sin 2x} \cdot \frac{1}{\sqrt{\sin 2x}}} \Rightarrow \sqrt{\sin 2x} + \frac{1}{\sqrt{\sin 2x}} \geq 2\right)$$

Now, (I)+(IV) $\Rightarrow \sin x + \cos x + \tan x + \cos x + \frac{1}{2}(\sec x + \csc x) \geq 2(1 + \sqrt{2})$

Equality holds for $x = \frac{\pi}{4} \therefore x = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$ *is the solution of the given equation.*

711. **Solve for real numbers:**

$$\sqrt[4]{x^4+16x^3+49x^2+81}+\sqrt[3]{x^3+25x^2+27}=\sqrt{4x^3+25x^2+100x+36}$$

Proposed by Naren Bhandari-Bajura- Nepal

Solution by Michael Sterghiou-Greece

$$\sqrt[4]{x^4+16x^3+49x^2+81}+\sqrt[3]{x^3+25x^2+27}=\sqrt{4x^3+25x^2+100x+36}$$

(1) can be written as $\sqrt[4]{A}+\sqrt[3]{B}\neq\sqrt{C}$ where A,B,C the respective polynomial roots.

1) Assume $x>0$ then all roots have meaning. We have $(A,B,C>0)$

$\sqrt[3]{B}>\sqrt[4]{A}\leftrightarrow B^4-A^3>0\leftrightarrow x^2\cdot f(x)>0$ *where* $f(x)=52x^9+2835x^8+53808x^7+$

$+353647x^6+79476x^5+1488x^4-162324x^3+2130624x^2-236196x+1033833$

$f(x)>0$ *for* $x>0$ *as can be easily shown [if $x\geq 1$ obvious, if $x<1$ the constant overweight the negative terms]*

$\sqrt{C}>2\sqrt[3]{B}\leftrightarrow C^3>64B^2\leftrightarrow C^3-64B^2>0$, *because* $C^3-64B^2=x\cdot g(x)$ *where*

$g(x)=64x^8+1200x^7+12300x^6+77289x^5+325900x^4+863900x^3+$

$+1552096x^2+1090800x+388800>0$

Therefore $\sqrt{C}>2\sqrt[3]{B}>\sqrt[3]{B}+\sqrt[3]{A}$ so, (1) has no solutions.

2) Assume $x<0$. The inequalities $A\geq 0, C\geq 0$ are true when $x\geq\vartheta$ where $\vartheta\simeq -0.4$ so $-|\vartheta|\leq x<0$ in which $B>0$ too. The above polynomial $f(x)$ as positive (easy as the negative terms – powers of $9,7,5$ are smaller than the constant term).

We can also show that $\sqrt{C}<2\sqrt[4]{A}\leftrightarrow C^2-16A<0\leftrightarrow x\cdot h(x)<0$

$h(x)=16x^5+200x^4+1409x^3+5032x^2+11016x+7200$ *as all negative terms are less than constant for $x=-0.4$ in which they become maximal. Now,*

$\sqrt{C}<2\sqrt[4]{A}<\sqrt[4]{A}+\sqrt[3]{B}$ *hence no solution. Therefore, the only solution is $x=0$. Done.*

712. **Compute the integral in a closed – form**

$$\int_0^\infty \log\left(\frac{\sqrt{5}-1}{2}-e^{-\pi x}\right)e^{-\pi x}dx$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution 1 by Dawid Bialek-Poland

$$\int_0^\infty \log\left(\frac{\sqrt{5}-1}{2}-e^{-\pi x}\right)e^{-\pi x}dx = \left| \begin{array}{c} t=\frac{\sqrt{5}-1}{2}-e^{-\pi x} \Rightarrow \frac{\sqrt{5}-3}{2} \le t \le \frac{\sqrt{5}-1}{2} \\ dx = \frac{dt}{\pi e^{-\pi x}} \end{array} \right| =$$

$$= \frac{1}{\pi}\int_{\frac{\sqrt{5}-3}{2}}^{\frac{\sqrt{5}-1}{2}} \ln(t)\,dt \overset{I.B.P}{=} \frac{1}{\pi}[t\ln(t)-t]_{\frac{\sqrt{5}-3}{2}}^{\frac{\sqrt{5}-1}{2}} =$$

$$= \frac{1}{\pi}\left[\left(\frac{\sqrt{5}-1}{2}\right)\ln\left(\frac{\sqrt{5}-1}{2}\right)-\left(\frac{\sqrt{5}-1}{2}\right)-\left(\frac{\sqrt{5}-3}{2}\right)\ln\left(\frac{\sqrt{5}-3}{2}\right)+\left(\frac{\sqrt{5}-3}{2}\right)\right] =$$

$$= \frac{1}{\pi}\left[\left(\frac{\sqrt{5}-1}{2}\right)\ln\left(\frac{\sqrt{5}-1}{2}\right)-\left(\frac{\sqrt{5}-3}{2}\right)\ln\left(\frac{\sqrt{5}-3}{2}\right)-1\right] \quad (1)$$

We can also express the result (1) using the golden ratio:

Let: $\frac{\sqrt{5}-1}{2}=\Phi-1$ $\frac{\sqrt{5}-3}{2}=\Phi-2$, **where** $\Phi=\frac{\sqrt{5}+1}{2}$ **– the golden ratio. Then, we get:**

$$\int_0^\infty \log\left(\frac{\sqrt{5}-1}{2}-e^{-\pi x}\right)e^{-\pi x}dx = \frac{1}{\pi}[(\Phi-1)\ln(\Phi-1)-(\Phi-2)\ln(\Phi-2)-1] =$$

$$= \frac{1}{\pi}\big[(\Phi-1)\ln(\Phi-1)-(\Phi-2)\ln\big((-1)\cdot(2-\Phi)\big)-1\big] =$$

$$= \frac{1}{\pi}[(\Phi-1)\ln(\Phi-1)-(\Phi-2)\ln(2-\Phi)-1-(\Phi-2)\ln(-1)] \overset{\ln(-1)=i\pi}{=}$$

$$= \frac{1}{\pi}[(\Phi-1)\ln(\Phi-1)-(\Phi-2)\ln(2-\Phi)-1]-i(\Phi-2) =$$

$$= \underbrace{\frac{1}{\pi}\left[\Phi\ln\left(\frac{\Phi-1}{2-\Phi}\right)+\ln\left(\frac{(2-\Phi)^2}{\Phi-1}\right)-1\right]}_{Real\ part} - \underbrace{i(\Phi-2)}_{Imaginary\ part}$$

Solution 2 by Tobi Joshua-Nigeria

$$\int_0^\infty \log\left(\frac{\sqrt{5}-1}{2}-e^{-\pi x}\right)e^{-\pi x}dx$$

$$\frac{1}{\pi}\int_0^1 \log\left(\frac{\sqrt{5}-1}{2}-x\right)dx$$

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Let $\phi=\frac{\sqrt{5}-1}{2}$ ***and*** $1-\frac{1}{\phi}=\frac{\sqrt{5}-3}{\sqrt{5}-1}$

$$\frac{1}{\pi}\int_{0}^{1}\log(\phi-x)\,dx$$

$$\frac{1}{\pi}\int_{0}^{1}\log\left(1-\frac{x}{\phi}\right)dx+\frac{\log\phi}{\pi}$$

$$-\frac{1}{\pi}\int_{0}^{1}x^{k+1}\,dx\sum_{k=0}^{\infty}\frac{\left(\frac{1}{\phi}\right)^{k+1}}{(k+1)}+\frac{\log\phi}{\pi}$$

$$-\frac{1}{\pi}\int_{0}^{1}x^{k+1}\,dx\sum_{k=0}^{\infty}\frac{\left(\frac{1}{\phi}\right)^{k+1}}{(k+1)}+\frac{\log\phi}{\pi}$$

$$-\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\left(\frac{1}{\phi}\right)^{k}}{(k)(k+1)}+\frac{\log\phi}{\pi}$$

$$-\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\left(\frac{1}{\phi}\right)^{k}}{(k)}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\left(\frac{1}{\phi}\right)^{k}}{(k+1)}+\frac{\log\phi}{\pi}$$

$$-\frac{1}{\pi}Li_1\left(\frac{1}{\phi}\right)+\frac{1}{\pi}\left[-1+\phi Li_1\left(\frac{1}{\phi}\right)\right]+\frac{\log\phi}{\pi}$$

$$\frac{1}{\pi}\log\left(1-\left(\frac{1}{\phi}\right)\right)+\frac{1}{\pi}\left[-1-\phi\log\left(1-\left(\frac{1}{\phi}\right)\right)\right]+\frac{\log\phi}{\pi}$$

$$\log\left(1-\left(\frac{1}{\phi}\right)\right)\left[\frac{1}{\pi}-\frac{\phi}{\pi}\right]-\frac{1}{\pi}+\frac{\log\phi}{\pi};\ \log\left(\frac{\sqrt{5}-3}{\sqrt{5}-1}\right)\left[\frac{\left(\frac{3-\sqrt{5}}{2}\right)}{\pi}\right]-\frac{1}{\pi}+\frac{\log\left(\frac{\sqrt{5}-1}{2}\right)}{\pi}.\ \textit{But}$$

$$\log\left(\frac{\sqrt{5}-3}{\sqrt{5}-1}\right)=\log\left(\frac{\sqrt{5}-1}{2}\right)+i\pi\quad\oplus$$

$$\frac{\left[\log\left(\frac{\sqrt{5}-1}{2}\right)+i\pi\right]}{\pi}\left[\left(\frac{3-\sqrt{5}}{2}\right)\right]-\frac{1}{\pi}+\frac{\log\left(\frac{\sqrt{5}-1}{2}\right)}{\pi}$$

$$\frac{\left[\log\left(\frac{\sqrt{5}-1}{2}\right)\right]}{\pi}\left[\left(\frac{3-\sqrt{5}}{2}\right)\right]+\frac{\log\left(\frac{\sqrt{5}-1}{2}\right)}{\pi}-\frac{1}{\pi}+i\left[\left(\frac{3-\sqrt{5}}{2}\right)\right]$$

$$\frac{\left[\log\left(\frac{\sqrt{5}-1}{2}\right)\right]}{\pi\left(\sqrt{5}+1\right)}\left(\sqrt{5}+1\right)\left[\left(\frac{3-\sqrt{5}}{2}\right)\right]+\frac{\left(\sqrt{5}+1\right)\log\left(\frac{\sqrt{5}-1}{2}\right)}{\pi\left(\sqrt{5}+1\right)}-\frac{1}{\pi}+i\left[\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right)\right]$$

$$\frac{\left[\log\left(\frac{\sqrt{5}-1}{2}\right)\right]}{\pi\left(\sqrt{5}+1\right)}\left[\left(\sqrt{5}-1\right)\right]+\frac{\left(\sqrt{5}+1\right)\log\left(\frac{\sqrt{5}-1}{2}\right)}{\pi\left(\sqrt{5}+1\right)}-\frac{1}{\pi}+i\left[\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right)\right]$$

$$\frac{\left[\sqrt{5}\log\left(\frac{-2\sqrt{5}+6}{4}\right)+\log\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right)\right]}{\pi\left(\sqrt{5}+1\right)}-\frac{1}{\pi}+i\left[\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right)\right]$$

$$\frac{\left[\sqrt{5}\log\left(\frac{\left(3-\sqrt{5}\right)}{2}\right)\right]}{\pi\left(\sqrt{5}+1\right)}-\frac{1}{\pi}+i\left[\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right)\right]$$

$$\frac{\left[\sqrt{5}\log\left(\frac{2}{3+\sqrt{5}}\right)\right]}{\pi\left(\sqrt{5}+1\right)}-\frac{1}{\pi}+i\left[\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right)\right]$$

i = ***imaginary number***

713. **Find:**

$$\int_{-\frac{\pi}{4}}^{-\frac{\pi}{4}}\frac{\sec^6 x-\tan^6 x}{(a^x+1)\cos^2 x}dx,\ (a>0)$$

Proposed by Artan Ajredini-Presheva-Serbie

Solution 1 by Avishek Mitra-West Bengal-India

$$I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\sec^6 x-\tan^6 x}{(a^x+1)\cdot\cos^2 x}dx=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\sec^6(-x)-\tan^6(-x)}{(a^{-x}+1)\cdot\cos^2(-x)}dx$$

$$=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{a^x(\sec^6 x-\tan^6 x)}{(a^x+1)\cos^2 x}dx \Rightarrow I+I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{(a^x+1)(\sec^6 x-\tan^6 x)}{(a^x+1)\cos^2 x}dx$$

$$\Rightarrow 2I=2\int_{0}^{\frac{\pi}{4}}\frac{(\sec^6 x-\tan^6 x)}{\cos^2 x}dx$$

$$\Rightarrow I=\int_{0}^{\frac{\pi}{4}}\frac{(\sec^2 x-\tan^2 x)(\sec^4 x+\sec^2 x\cdot\tan^2 x+\tan^4 x)}{\cos^2 x}dx$$

$$\Rightarrow I=\int_{0}^{\frac{\pi}{4}}\frac{(\sec^2 x-\tan^2 x)^2+2\sec^2 x\cdot\tan^2 x+\sec^2 x\cdot\tan^2 x}{\cos^2 x}dx$$

$$\Rightarrow I=\int_{0}^{\frac{\pi}{4}}\frac{(1-3\sec^2 x\cdot\tan^2 x)}{\cos^2 x}dx\Rightarrow I=\int_{0}^{\frac{\pi}{4}}\sec^2 x\,dx+3\int_{0}^{\frac{\pi}{4}}\tan^2 x\cdot\sec^4 x\,dx$$

$$=[\tan x]_0^{\frac{\pi}{4}}+3\int_{0}^{\frac{\pi}{4}}\tan^2 x\,(\tan^2 x+1)\cdot\sec^2 x\,dx$$

$$=1+3\int_{0}^{\frac{\pi}{4}}\tan^4 x\,d(\tan x)+3\int_{0}^{\frac{\pi}{4}}\tan^2 x\,d(\tan x)$$

$$=1+3\cdot\left[\frac{\tan^5 x}{5}\right]_0^{\frac{\pi}{4}}+3\left[\frac{\tan^3 x}{3}\right]_0^{\frac{\pi}{3}}=1+3\cdot\frac{1}{5}+3\cdot\frac{1}{3}=\frac{13}{5}$$

Solution 2 by Kartick Chandra Betal-India

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\sec^6 x-\tan^6 x}{(a^x+1)\cos^2 x}dx=\int_{0}^{\frac{\pi}{4}}\frac{(\sec^6 x-\tan^6 x)}{\cos^2 x}\left[\frac{1}{a^x+1}+\frac{a^x}{1+a^x}\right]dx=\int_{0}^{\frac{\pi}{4}}\frac{(\sec^6 x-\tan^6 x)}{\cos^2 x}dx=$$

$$=\int_{0}^{\frac{\pi}{4}}\frac{(\sec^2 x-\tan^2 x)^3+3\sec^2 x\tan^2 x\,(\sec^2 x-\tan^2 x)}{\cos^2 x}dx$$

$$=\int_0^{\frac{\pi}{4}}\frac{1+\sec^2 x\tan^2 x}{\cos^2 x}dx=\int_0^{\frac{\pi}{4}}\frac{\sec^2 x+3\sec^2 x(1+\tan^2 x)\tan^2 x}{1}dx$$

$$=\int_0^{\frac{\pi}{4}}(\sec^2 x+3\tan^2 x\sec^2 x+3\tan^4 x\sec^2 x)\,dx$$

$$=\left[\tan x+3\frac{\tan^3 x}{3}+3\frac{\tan^5 x}{5}\right]_0^{\frac{\pi}{4}}=1+1+\frac{3}{5}=\frac{13}{5}$$

Solution 3 by Sohini Mondal-India

$$I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\sec^6 x-\tan^6 x}{(a^x+1)\cos^2 x}dx=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\sec^6(-x)-\tan^6(-x)}{(a^{-x}+1)\cos^2(-x)}dx$$

$$=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{a^x(\sec^6 x-\tan^6 x)}{(a^x+1)\cos^2 x}dx\Rightarrow 2I=2\int_0^{\frac{\pi}{4}}\frac{(a^x+1)(\sec^6 x-\tan^6 x)dx}{(a^x+1)\cos^2 x}$$

$$\Rightarrow I=\int_0^{\frac{\pi}{4}}[(\sec^2 x)^3-\tan^6 x]\sec^2 x\,dx=\int_0^{\frac{\pi}{4}}[(\tan^2 x+1)^3-\tan^6 x]\sec^2 x\,dx$$

$$=\int_0^{\frac{\pi}{4}}(\tan^6 x+3\tan^4 x+3\tan^2 x+1-\tan^6 x)\sec^2 x\,dx$$

$$=\int_0^{\frac{\pi}{4}}(3\tan^4 x+3\tan^2 x)\sec^2 x\,dx+\int_{\frac{\pi}{4}}^{\frac{\pi}{4}}\sec^2 x\,dx=\int_0^1(3z^4+3z^2)dz+[\tan x]_0^{\frac{\pi}{4}}$$

[Let** $z=\tan x\,,dz=\sec^2 x\,dx$**]

$$=3\left[\frac{z^5}{5}+\frac{z^3}{3}\right]_0^1+1=\frac{3}{5}+\frac{3}{3}+1=\frac{3}{5}+2$$

$I=\frac{13}{5}$ ***[Answer]***

Solution 4 by Suhash Maurya-India

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2 x - \tan^6 x}{(a^x+1)\cos^2 x} dx, (a > 0) \quad \textbf{(1)}$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^6 x - \tan^6 x}{(a^{-x}+1)\cos^2 x} dx, \int_a^b f(x)\, dx = \int_{-a}^{b} f(a+b-x)\, dx$$

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(\sec^6 x - \tan^6 x)a^x}{(1+a^x)\cos^2 x} dx \quad \textbf{(2)}$$

Add (1) and (2) we get: $2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^6 x - \tan^6 x}{\cos^2 x} dx$

$$2I = 2\int_0^{\frac{\pi}{4}} \sec^8 x\, dx - 2\int_0^{\frac{\pi}{4}} \tan^6 x \sec^2 x\, dx$$

$$I = \int_0^{\frac{\pi}{4}} (1+\tan^2 x)^3 \sec^2 x\, dx - \int_0^1 t^6 dt$$

$$= \int_0^1 (1+t^2)^3\, dt - \left[\frac{t^7}{7}\right]_0^1 = \int_0^1 (1+t^6+3t^2+3t^4) dt - \frac{1}{7}$$

$$= \left[t + \frac{t^7}{7} + t^3 + \frac{3}{5} + 5\right]_0^1 - \frac{1}{7} = 1 + \frac{1}{7} + 1 + \frac{3}{5} - \frac{1}{7} = 2 + \frac{3}{5} = \frac{13}{5}$$

$$I = \frac{13}{5}$$

Solution 5 by Tobi Joshua-Nigeria

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^6 x - \tan^6 x}{(a^x+1)\cos^2 x} dx \quad \textbf{(1)}$$

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^6 x - \tan^6 x}{(a^{-x}+1)\cos^2 x} dx \quad \textbf{(2)}$$

$$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^6 x - \tan^6 x}{\cos^2 x} dx$$

$$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left((1+\tan^2 x)^{\frac{6}{2}} - \tan^6 x\right) d(\tan x)$$

$$2I = 2\int_0^{\frac{\pi}{4}} \left((1+\tan^2 x)^{\frac{6}{2}} - \tan^6 x\right) d(\tan x) \ \textit{(Even function)}$$

$$I = \int_0^1 ((1+t^2)^3 - t^6)\, dt$$

$$I = \int_0^1 (1 + 3t^2 + 3t^4 + t^6 - t^6)\, dt$$

$$I = \int_0^1 (1 + 3t^2 + 3t^4)\, dt$$

$$I = \left[t + t^3 + \frac{3t^5}{5}\right]_0^1$$

$$I = \left[\frac{13}{5}\right]$$

714. **If we define for any complex number m**

$$f(m,n) = \int_{e^{-\pi n}}^{\infty} e^{-mx^2}\, dx,\ Re\,(m) > 0$$

then evaluate the integral

$$\int_{-\infty}^{\infty} f(m,n)\, e^{-\pi mn}\, dn$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Abdul Hafeez-Ayinde-Nigeria

$$\Omega = \int_{e^{-\pi n}}^{\infty} e^{-mx^2}\, dx$$

$$\Omega = \int_0^\infty e^{-mx^2}\,dx - \int_0^{e^{-\pi n}} e^{-mx^2}\,dx$$

$$\Omega = \int_0^\infty e^{-mx^2}\,dx - \frac{1}{\sqrt{m}}\int_0^{\sqrt{m}e^{-\pi n}} e^{-x^2}\,dx$$

$$\Omega = \frac{1}{\sqrt{m}}\int_0^\infty e^{-x^2}\,dx - \frac{1}{\sqrt{m}}\cdot\frac{\sqrt{\pi}}{2}\operatorname{erf}\left(\sqrt{m}e^{-\pi n}\right)$$

$$\Omega = \frac{1}{\sqrt{m}}\cdot\frac{\sqrt{\pi}}{2} - \frac{1}{\sqrt{m}}\cdot\frac{\sqrt{\pi}}{2}\operatorname{erf}\left(\sqrt{m}e^{-\pi n}\right)$$

$$\Omega = \frac{1}{\sqrt{m}}\cdot\frac{\sqrt{\pi}}{2}\operatorname{erfc}\left(\sqrt{m}e^{-\pi n}\right)$$

$$\int_{-\infty}^{\infty}\Omega\, e^{-\pi m n}dn = \Lambda = \frac{1}{\sqrt{m}}\cdot\frac{\sqrt{\pi}}{2}\int_{-\infty}^{\infty}\operatorname{erfc}\left(\sqrt{m}e^{-\pi n}\right)e^{-\pi m n}dn$$

$$u = \sqrt{m}e^{-\pi n},\ du = -\pi u\,dn$$

$$\Lambda = \frac{1}{\sqrt{m}}\cdot\frac{\sqrt{\pi}}{2}\cdot\frac{1}{\pi\cdot m^{\frac{m}{2}}}\int_0^\infty \frac{u^m}{u}\operatorname{erfc}(u)\,du$$

$$\Lambda = \frac{1}{\sqrt{m}}\cdot\frac{\sqrt{\pi}}{2}\cdot\frac{1}{\pi m^{\frac{m}{2}}}\int_0^\infty u^{m-1}\operatorname{erfc}(u)\,du$$

$$\Lambda = \frac{1}{\sqrt{m}}\cdot\frac{\sqrt{\pi}}{2}\cdot\frac{1}{\pi\cdot m^{\frac{m}{2}}}\left(\left|u^m\operatorname{erfc}(u)\right|_0^\infty - (m-1)\int_0^\infty \operatorname{erfc}(u)\,u^{m-1} + \frac{2}{\sqrt{\pi}}\int_0^\infty u^m e^{-\pi^2}\,du\right)$$

$$\Lambda = \frac{1}{\sqrt{m}}\cdot\frac{\sqrt{\pi}}{2}\cdot\frac{1}{\pi\cdot m^{\frac{m}{2}}}\left(\frac{2}{m\sqrt{\pi}}\int_0^\infty u^m e^{-u^2}du\right)$$

$$\Lambda = \frac{1}{\sqrt{m}}\cdot\frac{\sqrt{\pi}}{2}\cdot\frac{1}{\pi\cdot m^{\frac{m}{2}}}\left(\frac{1}{m\sqrt{\pi}}\int_0^\infty u^{\frac{m-1}{2}}e^{-u}du\right)$$

$$\Lambda = \frac{1}{\sqrt{m}}\cdot\frac{\sqrt{\pi}}{2}\cdot\frac{1}{\pi\cdot m^{\frac{m}{2}}}\left(\frac{1}{m\sqrt{\pi}}\Gamma\left(\frac{m+1}{2}\right)\right);\ \Lambda = \frac{1}{2\pi m^{\frac{m+3}{2}}}\Gamma\left(\frac{m+1}{2}\right)$$

715. Find:

$$\Omega(a)=\int_0^1\left(\frac{a^x+3+(3-a)^x}{(3+a^x-a)(a+(3-a)^x)}\right)dx,\ 1<a<2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Samir Haj Ali-Damascus-Syria

$$\Omega(a)=\int_0^1\frac{a^x+3+(3-a)^x}{(3+a^x-a)(a+(3-a)^x)}dx;\ 1<a<2$$

$$=\int_0^1\frac{a^x+3-a+(3-a)^x+a}{(3+a^x-a)(a+(3-a)^x)}dx=\int_0^1\frac{dx}{a+(3-a)^x}+\int_0^1\frac{dx}{3+a^x-a}=I_1+I_2$$

Now: $I_1=\int_0^1\frac{dx}{a+(3-a)^x}$ *let* $(3-a)^x=t$

$$I_1=\int_1^{3-a}\frac{dt}{(a+t)(t\cdot\ln(3-a))}$$

$$=\frac{1}{\ln(3-a)}\int_0^{3-a}\frac{dt}{t(t+a)}=\frac{1}{a\cdot\ln(3-a)}\cdot\ln\left(\frac{1}{3}(3-a)(1+a)\right)$$

$$I_2=\int_0^1\frac{dx}{3+a^x-a};t=a^x$$

$$=\int_1^a\frac{dt}{(3-a+t)t\cdot\ln a}=\frac{1}{(a-3)\cdot\ln a}\cdot\ln\left(\frac{3}{a(4-a)}\right)$$

$$\Rightarrow\Omega(a)=\frac{\ln\left(\frac{1}{3}(3-a)(1+a)\right)}{a\cdot\ln(3-a)}+\frac{\ln\left(\frac{3}{a(4-a)}\right)}{(a-3)\cdot\ln a}$$

Solution 2 by Surjeet Singhania-India

$$\Omega(a)=\int_0^1\frac{a^x+3+(3-a)^x}{(3+a^x-a)(a+(3-a)^x)}dx,\ 1<a<2$$

$$\Omega(a)=\int_0^1\frac{(a^x+3-a)+[a+(3-a)^x]}{(3+a^x-a)(a-1(3-a)^x)}dx$$

$$\Omega(a)=\int_0^1\frac{1}{(a+(3-a)^x)}dx+\int_0^1\frac{1}{a^x-a+3}dx$$

$$\Omega(a)=I_1+I_2\ \ \textbf{(1)}$$

$$I_1=\int_0^1\frac{1}{[a+(3-a)^x]}dx=\int_0^1\frac{(3-a)^x}{(3-a)^x(a+(3-a)^x)}dx$$

Putting $(3-a)^x=y$

$$x=\frac{1}{\ln(3-a)}\ln y$$

$$dx=\frac{1}{y\ln(3-a)}dy$$

As $x=0;y=$

As $x=1;y=3-a$

$$=\int_1^{3-a}\frac{1}{(a+y)}\times\frac{1}{y\ln(3-a)}dy=\frac{1}{\ln(3-a)}\int_0^{3-a}\frac{1}{y(y+a)}dy$$

$$=\frac{1}{\ln(3-a)}\int_0^{3-a}\left(\frac{-1}{a(y+a)}+\frac{1}{a}\cdot\frac{1}{y}\right)dy=\frac{1}{\ln(3-a)}\left(\frac{1}{a}\right)\ln\left(\frac{y}{y+a}\right)\Bigg]_1^{3-a}$$

$$=\frac{1}{a\ln(3-a)}\left[\ln\left(\frac{3-a}{3}\right)-\ln\left(\frac{1}{1+a}\right)\right]=\frac{1}{a\ln(3-a)}\ln\left(\frac{(3-a)(1+a)}{3}\right)$$

$$I_1=\frac{1}{a\ln(3-a)}\ln\left(\frac{3+2a-a^2}{3}\right)$$

$$I_2=\int_0^1\frac{1}{(a^x-a+3)}dx$$

Putting $a^x=y;\ x=\frac{1}{\ln a}\ln(y);\ dy=\frac{1}{y\ln(a)}dy$

$$I=\int_1^a\frac{1}{(y-a+3)y(\ln a)}dy$$

$$=\frac{1}{\ln(a)}\int_1^a\frac{1}{y(y-a+3)}dy=\frac{1}{\ln(a)\,(a-3)}\int_1^a\frac{1}{y-a+3}dy+$$

$$+\frac{1}{\ln(a)\,(3-a)}\int_1^a\frac{1}{y}dy=\frac{1}{\ln(a)\,(3-a)}[\ln y]_1^a-\ln(y-2+3)]_1^a]$$

$$=\frac{1}{\ln(a)\,(3-a)}[\ln a-\ln(3)+\ln(4-a)]=\frac{1}{\ln(a)\,(3-a)}\ln\left(\frac{(4-a)a}{3}\right)$$

Hence $\Omega(a)=\frac{1}{a\ln(3-a)}\ln\left[\frac{3+2a-a^2}{3}\right]+\frac{1}{(3-a)\ln(a)}\ln\left[\frac{(4-a)a}{3}\right]$

716. Integrate:

$$\int\frac{\ln\left(1+\sqrt{1-4x}\right)}{x}dx$$

Proposed by Prem Kumar-India

Solution 1 by Avishek Mitra-West Bengal-India

$$\Rightarrow \textit{Let } \sqrt{1-4x}=z\Rightarrow 1-4x=z^2\Rightarrow -4dx=2z\,dz\Rightarrow dx=-\frac{zdz}{2}$$

$$\Rightarrow\frac{dx}{x}=-\frac{1}{2}\cdot\frac{z\,dz}{\frac{(1-z^2)}{4}}=-\frac{2z\,dz}{(1-z^2)}\Leftrightarrow\Omega=-2\int\frac{z\log(1+z)}{(1-z^2)}dz$$

$$\Rightarrow z=\sin y\Rightarrow dz=\cos y\,dy$$

$$\Omega=-2\int\frac{\sin y\cdot\log(1+\sin y)}{(1-\sin^2 y)}\cos y\,dy=-2\int\tan y\cdot\log(1+\sin y)\,dy$$

$$=-2\left[\log(1+\sin y)\cdot\sec^2 y-\int\frac{\cos y\cdot\sec^2 y}{(1+\sin y)}\right]dy$$

$$=-2\log(1+\sin y)\cdot\sec^2 y+2\int\frac{dy}{\cos y\,(1+\sin y)}$$

$$\Leftrightarrow I_1=\int\frac{dy}{\cos y\cdot(1+\sin y)}\Rightarrow\sin y=u\Rightarrow\cos y\,dy=du\Rightarrow\frac{dy}{\cos y}=\frac{du}{1-u^2}$$

$$=\int\frac{du}{(1+u)(1-u^2)}=\int\frac{du}{(1+u)^2(1-u)}$$

$$=\frac{1}{2}\left[\int\frac{(1+u)+(1-u)}{(1+u)^2(1-u)}\right]du=\frac{1}{2}\left[\int\frac{du}{(1-u^2)}+\int\frac{du}{(1+u)^2}\right]$$

$$=\frac{1}{2}\left[\int\frac{\cos y\,dy}{(1-\sin^2 y)}-\frac{1}{(1+u)}\right]$$

$$=\frac{1}{2}\left[\int\sec y\,dy-\frac{1}{(1+\sin y)}\right]=\frac{1}{2}\left[\log|(\sec y+\tan y)|-\frac{1}{(1+\sin y)}\right]$$

$$\Omega=-2\log(1+\sin y)\cdot\sec^2 y+\log|(\sec y+\tan y)|-\frac{1}{(1+\sin y)}+c$$

$$=-2\frac{\log(1+z)}{(1-z^2)}+\log\left|\frac{(1+z)}{\sqrt{1-z^2}}\right|-\frac{1}{(1+z)}+c$$

$$=-2\cdot\frac{\log\left(1+\sqrt{1-4x}\right)}{1-1+4x}+\log\left(\frac{1+\sqrt{1-4x}}{\sqrt{1-1+4x}}\right)-\frac{1}{1+\sqrt{1-4x}}+c$$

$$=\log\left|\left(\frac{1+\sqrt{1-4x}}{2\sqrt{x}}\right)\right|-\frac{1}{1+\sqrt{1-4x}}-\frac{\log\left(1+\sqrt{1-4x}\right)}{2x}+c$$

Solution 2 by Surjeet Singhania-India

$$I=\int\frac{\ln\left(1+\sqrt{1-4x}\right)}{x}dx$$

$1-4x>0;\ 1>4x;\ 4x<1;\ x<\frac{1}{4};$ ***Let*** $y=1+\sqrt{1-4x}$

$$y-1=\sqrt{1-4x}\Rightarrow(y-1)^2=1-4x$$

$$4x=1-(y-1)^2;\ x=\frac{1-(y-1)^2}{4}$$

$$dx=\frac{1}{4}[-2(y-1)]dy;\ dx=\left(\frac{1-y}{2}\right)dy$$

$$I=\int\frac{\ln y}{\frac{1-(y-1)^2}{4}}\times\frac{(1-y)}{2}dy;\ I=2\int\frac{(1-y)\ln y}{(1-y+1)(1+y-1)}dy$$

$$I=2\int\frac{(1-y)\ln y}{(2-y)y}dy.\ \textit{So, } I=2\int\frac{\ln y}{2y}dy+\frac{2}{2}\int\frac{\ln y}{y-2}dy$$

$$I=\int\frac{\ln y}{y}dy+\int\frac{\ln y}{y-2}dy;\ I=\frac{\ln^2 y}{2}+I_1$$

$$I_1=\int\frac{\ln y}{y-2}dy=\frac{1}{2}\int\frac{\ln y}{\frac{y}{2}-1}dy$$

$$\textit{Put } \frac{y}{2}-1=\gamma \Rightarrow d\gamma=\frac{dy}{2}$$

$$\frac{y}{2}=\gamma+1 \Rightarrow y=2(\gamma+1)$$

$Li_n(x)$ *is poly logarithm*

$$\ln(1-x)=-Li_1(x)$$

$$I_1=\int\frac{\ln\big((2)(\gamma+1)\big)}{\gamma}d\gamma$$

$$I_1=\int\frac{\ln 2}{\gamma}d\gamma+\int\frac{\ln(\gamma+1)}{\gamma}d\gamma$$

$$I_1=\ln 2\ln\gamma+\left(-\int\frac{Li_1(-\gamma)}{\gamma}d\gamma\right)$$

$$I_1=\ln 2\ln\gamma-Li_2(-x)+c$$

$$\textit{So } I=\frac{\ln^2 y}{2}+\ln 2\ln\left(\frac{y}{2}-1\right)-Li_2\left(1-\frac{y}{2}\right)+c$$

$$I=\frac{\ln^2\left(1+\sqrt{1-4x}\right)}{2}+\ln(2)\ln\left(\frac{\sqrt{1-4x}-1}{2}\right)-Li_2\left(\frac{1-\sqrt{1-4x}}{2}\right)+c$$

717. If we can express the integral

$$\frac{\pi}{12}+\int_0^{\infty}\log(\tanh(e^{-\pi x}))\,e^{-\pi x}dx$$

in the form

$$\frac{Li_2(1-t)+Li_2(-t)+\log(t+1)\log(t)}{2\pi}$$

then find the value of t.

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Feti Sinani-Kosovo

$$\textit{Let be } I(x)=\int\log(\tanh(e^{-\pi x}))\,e^{-\pi x}dx$$

$$\textit{sub. } \tanh(e^{-\pi x})=t \Rightarrow -\pi(1-t^2)e^{-\pi x}dx=dt \Rightarrow e^{-\pi x}dx=-\frac{dt}{\pi(1-t^2)} \Rightarrow I(x)=-\frac{1}{\pi}I(t)$$

$$I(t)=\int\frac{\log t}{1-t^2}dt=\frac{1}{2}\int\left(\frac{\log t}{1-t}+\frac{\log t}{1+t}\right)dt=\frac{1}{2}\left(Li_2(1-t)+Li_2(-t)+\log(t)\log(1+t)\right)$$

$$\int\frac{\log t}{1+t}dt=\int\log t\,d\log(1+t)=\log(t)\log(1+t)-\int\frac{\log(1+t)}{t}dt=$$

$$=\log(t)\log(1+t)+Li_2(-t)$$

$$I(x)=-\frac{Li_2(1-\tanh(e^{-\pi x}))+Li_2(-\tanh(e^{-\pi x}))+\log(\tanh(e^{-\pi x}))\log(1+\tanh(e^{-\pi x}))}{2\pi}$$

$$\int_0^{+\infty}\log(\tanh(e^{-\pi x}))\,e^{-\pi x}dx=$$

$$=\frac{Li_2(1-\tanh(1))+Li_2(-\tanh(1))+\log(\tanh(1))\log(1+\tanh(1))}{2\pi}-$$

$$-\frac{Li_2(1)}{2\pi}-\frac{1}{2\pi}\lim_{x\to+\infty}\log(\tanh(e^{-\pi x}))\log(1+\tanh(e^{-\pi x}))$$

$$\lim_{x\to+\infty}\log(\tanh(e^{-\pi x}))\log(1+\tanh(e^{-\pi x}))=\lim_{t\to+0}\log(t)\log(1+t)=$$

$$=\lim_{t\to+0}t\log(t)\cdot\frac{\log(1+t)}{t}=0$$

$$\int_0^{+\infty}\log(\tanh(e^{-\pi x}))\,e^{-\pi x}dx=$$

$$=\frac{Li_2(1-\tanh(1))+Li_2(-\tanh(1))+\log(\tanh(1))\log(1+\tanh(1))}{2\pi}-\frac{Li_2(1)}{2\pi}$$

$$\frac{\pi}{12}+\int_0^{+\infty}\log(\tanh(e^{-\pi x}))\,e^{-\pi x}dx=$$

$$=\frac{Li_2(1-\tanh(1))+Li_2(-\tanh(1))+\log(\tanh(1))\log(1+\tanh(1))}{2\pi}$$

718. Find:

$$\Omega=\int_0^{-1}\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\left(\sum_{k=1}^{\infty}\frac{x^k}{k}-\log\left(\frac{1}{1-x}\right)\right)\right)\right)dx$$

Proposed by Mohamed Arahman Jama-Somalia

Solution by Mokhtar Khassani-Mostaganem-Algerie

$$\Omega=\int_0^{-1}\left(\sum_{n=1}^{+\infty}\left(\frac{1}{n}\left(\sum_{k=1}^{n}\frac{x^k}{k}-\log\left(\frac{1}{1-x}\right)\right)\right)\right)dx=$$

$$=\int_0^{-1}\left(\sum_{n=1}^{+\infty}\left(\frac{1}{n}\left(\sum_{n=1}^{n}\int_0^x z^k dz-\log\left(\frac{1}{1-x}\right)\right)\right)\right)dx$$

$$=\int_0^{-1}\left(\sum_{n=1}^{+\infty}\left(\frac{1}{n}\left(\int_0^x\frac{1-z^n}{1-z}dz+\log(1-x)\right)\right)\right)dx=\int_0^{-1}\left(\sum_{n=1}^{+\infty}\left(\frac{1}{n}\left(\int_0^x\frac{z^n}{z-1}dz\right)\right)\right)dx$$

$$=\int_0^{-1}\left(\int_0^x\frac{1}{z-1}\sum_{n=1}^{+\infty}\frac{z^n}{n}dz\right)dx=\int_0^{-1}\left(-\int_0^x\frac{\log(1-z)}{z-1}dz\right)dx=$$

$$=\frac{1}{2}\int_{-1}^{0}\log^2(1-x)\,dx=\frac{1}{2}\int_1^2\log^2 x\,dx$$

$$=\frac{1}{2}\{x\log^2 x-2(x\log x-x)\}_1^2=(\log 2-1)^2\Rightarrow\Omega=(1-\log 2)^2$$

719. Find:

$$\Omega=\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\left(\int_\varepsilon^{1-\varepsilon}\left(\frac{(1-x^2)^2\log x}{1-x^6}\right)dx\right)$$

Proposed by Jesu Mhe-Nigeria

Solution 1 by Mokhtar Khassani-Mostaganem-Algerie

$$\Omega=\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_0^{1-\varepsilon}\frac{(1-x^2)^2\log x}{1-x^6}dx=\int_0^1\frac{(x^2-1)^2}{1-x^6}\log x\,dx=\int_0^1(x^2-1)^2\log x\sum_{n\ge 0}x^{6n}\,dx$$

$$=\sum_{n\ge 0}\int_0^1 x^{6n+4}\log x+x^{6n}\log x-2x^{6n+2}\log x\,dx$$

$$=\sum_{n\ge 0}\frac{2}{(6n+3)^2}-\frac{1}{(6n+5)^2}-\frac{1}{(6n+1)^2}$$

$$\int_0^1 x^k\log x\,dx=-\frac{1}{(k+1)^2}$$

$$=\frac{1}{36}\sum_{n\ge 0}\frac{2}{\left(n+\frac{1}{2}\right)^2}-\frac{1}{\left(n+\frac{5}{6}\right)^2}-\frac{1}{\left(n+\frac{1}{6}\right)^2}=\frac{1}{36}\left\{2\varphi_1\left(\frac{1}{2}\right)-\varphi\left(\frac{5}{6}\right)-\varphi_1\left(\frac{1}{6}\right)\right\}$$

$$=\frac{1}{36}\left\{2\cdot\frac{\pi^2}{2}-\left(\varphi_1\left(1-\frac{1}{6}\right)+\varphi_1\left(\frac{1}{6}\right)\right)\right\}=\frac{1}{36}\left\{\pi^2-\frac{\pi^2}{\sin^2\left(\frac{\pi}{6}\right)}\right\}=-\frac{\pi^2}{12}$$

$$\varphi_1(1-x)+\varphi_1(x)=\frac{\pi^2}{\sin^2(\pi x)}$$

Solution 2 by Abdul Mukhtar-Nigeria

$$I=\int_0^1\frac{(1-x^2)^2\ln x}{1-x^6}dx;\ I=\sum_{n=0}^{\infty}\int_0^1(1-x^2)^2x^{6n}\ln x\,dx$$

$$I=-\left[\sum_{n=0}^{\infty}\frac{1}{(6n+1)^2}+\sum_{n=0}^{\infty}\frac{1}{(6n+5)^2}\right]+2\sum_{n=0}^{\infty}\frac{1}{(6n+3)^2}$$

$$I=-\sum_{n=-\infty}^{\infty}\frac{1}{(6n+1)^2}+\frac{1}{9}\sum_{n=-\infty}^{\infty}\frac{1}{(2n+1)^2}$$

$$I=\frac{\pi}{36}Res\left(\frac{\cot(\pi s)}{\left(s+\frac{1}{6}\right)^2}\right)-\frac{\pi}{36}Res\left(\frac{\cot(\pi s)}{\left(s+\frac{1}{2}\right)^2}\right)$$

$I=\frac{\pi}{36}\frac{d}{ds}(\cot(\pi s))-\frac{\pi}{36}\frac{d}{ds}(\cot(\pi s))$ ***where*** $s=-\frac{1}{6}$ ***and*** $s=-\frac{1}{2}$

$$I=\frac{\pi}{36}(-4\pi)-\frac{\pi}{36}(-\pi);\ I=-\frac{4\pi^2}{36}+\frac{\pi^2}{36};\ I=\frac{-4\pi^2+\pi^2}{36}$$

$$I=\frac{-36\pi^2}{36}$$

$$I=-\frac{\pi^2}{12}$$

720. Find in terms of $a, b, 0 < a \le b < 2\pi$:

$$\Omega(a,b)=\int_a^b\int_a^b\int_a^b\left(\sin(x+y+z)-4\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{y+z}{2}\right)\sin\left(\frac{z+x}{2}\right)\right)dx\,dy\,dz$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Mokhtar Khassani-Mostaganem-Algerie

$$\int_a^b\int_a^b\int_a^b\left(\sin(x+y+z)-4\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x+z}{2}\right)\sin\left(\frac{z+y}{2}\right)\right)dx\,dy\,dz=$$

$$=\int_a^b\int_a^b\int_a^b(2\sin(x+y+z)-\sin x-\sin y-\sin z)\,dx\,dy\,dz$$

$$=\int_a^b\int_a^b(\cos(a+y+z)-\cos(b+y+z))dy\,dz-$$

$$-\int_a^b\int_a^b(\cos a-\cos b+(b-a)\sin y+(b-a)\sin z)\,dy\,dz$$

$$=2\int_a^b(\sin(a+b+z)-\sin(2a+z)+\sin(a+b+z)-\sin(2b+z))\,dz-$$

$$-\int_a^b\left((b-a)\cos a-(b-a)\cos b+(b-a)(\cos a-\cos b)+(b-a)^2\sin z\right)dz$$

$$=2\left(\begin{matrix}\cos(2a+b)+\cos(a+2b)+\cos(2a+b)-\cos(3a)+\cos(2a+b)-\\-\cos(a+2b)+\cos(3b)-\cos(a+2b)\end{matrix}\right)-$$

$$-\left((b-a)^2(\cos b-\cos a)+(b-a)^2(\cos b-\cos a)+(b-a)^2(\cos b-\cos a)\right)$$

$$=\frac{1}{16}\sin^3\left(\frac{b-a}{2}\right)\sin\left(3\frac{a+b}{2}\right)+3(b-a)^2(\cos b-\cos a)$$

Solution 2 by Avishek Mitra-West Bengal-India

$$\Leftrightarrow \sin x+\sin y+\sin z-\sin(x+y+z)$$

$$=2\sin\left(\frac{x+y}{2}\right)\cdot\cos\left(\frac{x-y}{2}\right)+2\cos\left(\frac{z+x+y+z}{2}\right)\sin\left(\frac{z-x-y-z}{2}\right)$$

$$= 2\sin\left(\frac{x+y}{2}\right)\cdot\cos\left(\frac{x-y}{2}\right) - 2\cos\left(\frac{x+y+2z}{2}\right)\sin\left(\frac{x+y}{2}\right)$$

$$= 2\sin\left(\frac{x+y}{2}\right)\left[\cos\left(\frac{x-y}{2}\right) - \cos\left(\frac{x+y+2z}{2}\right)\right]$$

$$= 2\sin\left(\frac{x+y}{2}\right)\cdot 2\sin\left(\frac{\frac{x-y}{2}+\frac{x+y+2z}{2}}{2}\right)\sin\left(\frac{\frac{x+y+2z}{2}-\frac{x-y}{2}}{2}\right)$$

$$= 4\sin\left(\frac{x+y}{2}\right)\cdot\sin\left(\frac{y+z}{2}\right)\cdot\sin\left(\frac{z+x}{2}\right)$$

$$\Leftrightarrow \sin(x+y+z) - 4\sin\left(\frac{x+y}{2}\right)\cdot\sin\left(\frac{y+z}{2}\right)\cdot\sin\left(\frac{z+x}{2}\right)$$

$$= \sin(x+y+z) - (\sin x + \sin y + \sin z) + \sin(x+y+z) =$$

$$= 2\sin(x+y+z) - (\sin x + \sin y + \sin z)$$

$$\Leftrightarrow I_2 = \int_a^b\int_a^b\int_a^b \sin(x+y+z)\,dx\,dy\,dz = -\int_a^b\int_a^b \{\cos(x+y+z)\}_a^b\,dydz$$

$$= -\int_a^b\int_a^b \{\cos(b+y+z) - \cos(a+y+z)\}\,dydz =$$

$$= -\left[\int_a^b\int_a^b \cos(b+y+z)\,dy\,dz - \int_a^b\int_a^b \cos(a+y+z)\,dy\,dz\right]$$

$$= -\left[\int_a^b \{\sin(b+y+z)\}_a^b\,dx - \int_a^b \{\sin(a+y+z)\}_a^b dz\right]$$

$$= -\left[\int_a^b \{\sin(2b+z) - \sin(b+a+z)\}dz - \int_a^b \{\sin(a+b+z) - \sin(2a+z)\}\,dz\right]$$

$$= -\left[\int_a^b \{\sin(2b+z) + \sin(2a+z)\}\,dz\right] = \left[\{\cos(2b+z)\}_a^b + \{\cos(2a+z)\}_a^b\right]$$

$$= \cos 3b - \cos(2b+a) + \cos(2a+b) - \cos 3a =$$

$$= (\cos 3b - \cos 3a) + (\cos(2a+b) - \cos(2b+a))$$

$$= 2\sin\frac{3(b+a)}{2}\cdot\sin\frac{3(a-b)}{2} + 2\cdot\sin\frac{3(a+b)}{2}\cdot\sin\frac{(b-a)}{2}$$

$$= 2\sin\frac{3(b+a)}{2}\left[2\sin\frac{\frac{3(a-b)}{2}+\frac{(b-a)}{2}}{2}\cdot\cos\frac{\frac{3(a-b)}{2}-\frac{(b-a)}{2}}{2}\right] =$$

$$= 4\sin\frac{3(b+a)}{2}\cdot\sin\left(\frac{a-b}{2}\right)\cos(a-b)$$

$$\Leftrightarrow I_2 = \int_a^b\int_a^b\int_a^b(\sin x+\sin y+\sin z)dx\,dy\,dz = 3\int_a^b\int_a^b\int_a^b \sin x\,dx\,dy\,dz$$

$$= 3\int_a^b\int_a^b\{\cos x\}_a^b\,dy\,dz = 3(\cos a-\cos b)\int_a^b\int_a^b dy\,dz = 3(b-a)^2(\cos a-\cos b)$$

$$\Leftrightarrow \Omega = \int_a^b\int_a^b\int_a^b\left\{\sin(x+y+z)-4\sin\left(\frac{x+y}{2}\right)\cdot\sin\left(\frac{y+z}{2}\right)\cdot\sin\left(\frac{z+x}{2}\right)\right\}dx\,dy\,dz$$

$$= 2\int_a^b\int_a^b\int_a^b \sin(x+y+z)\,dx\,dy\,dz - \int_a^b\int_a^b\int_a^b(\sin x+\sin y+\sin z)\,dx\,dy\,dz = 2I_1 - I_2$$

$$= 8\sin\frac{3(b+a)}{2}\sin\left(\frac{a-b}{2}\right)\cos(a-b) - 3(\cos a-\cos b)(b-a)^2$$

Solution 3 by Nelson Javier Villaherrera Lopez-El Salvador

$$\Omega(a,b) = \int_a^b\int_a^b\int_a^b\left[\sin(x+y+z)-4\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{y+z}{2}\right)\sin\left(\frac{z+x}{2}\right)\right]dx\,dy\,dz =$$

$$= \int_a^b\int_a^b\int_a^b\left[\sin(x+y+z)-4\sin\left(\frac{y+z}{2}\right)\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x+z}{2}\right)\right]dx\,dy\,dz$$

$$= \int_a^b\int_a^b\int_a^b\left[\sin(x+y+z)-4\sin\left(\frac{y+z}{2}\right)\frac{e^{\frac{j(x+y)}{2}}-e^{-\frac{f(x+y)}{2}}}{j2}\cdot\frac{e^{\frac{f(x+z)}{2}}-e^{-\frac{f(x+z)}{2}}}{j2}\right]dx\,dy\,dz$$

$$= \int_a^b\int_a^b\int_a^b\left[\sin(x+y+z)+2\sin\left(\frac{y+z}{2}\right)\frac{e^{\frac{f(2x+y+z)}{2}}-e^{\frac{f(y+z)}{2}}-e^{-\frac{f(y+z)}{2}}+e^{-\frac{f(2x+y+z)}{2}}}{2}\right]dx\,dy\,dz$$

$$= \int_a^b\int_a^b\int_a^b\left\{\sin(x+y+z)+2\sin\left(\frac{y+z}{2}\right)\left[\cos\left(x+\frac{y+z}{2}\right)-\cos\left(\frac{y-z}{2}\right)\right]\right\}dx\,dy\,dz$$

$$= \int_a^b \int_a^b \left\{-\cos(x+y+z) + 2\sin\left(\frac{y+z}{2}\right)\left[\sin\left(x+\frac{y+z}{2}\right) - x\cos\left(\frac{y-z}{2}\right)\right]\right\}_a^b dy\,dz$$

$$= \int_a^b \int_a^b \left\{\cos(a+y+z) - \cos(b+y+z) + 2\sin\left(\frac{y+z}{2}\right)\left[\sin\left(b+\frac{y+z}{2}\right) - \sin\left(a+\frac{y+z}{2}\right) - (b-a)\cos\left(\frac{y-z}{2}\right)\right]\right\} dy\,dz$$

$$\Omega(a,b) = \int_a^b \int_a^b \left\{\begin{matrix} \cos(a+y+z) - \cos(b+y+z) + \sin(b)\sin(y+z) + \\ + \cos(b)\,[1-\cos(y+z)] - \sin(a)\sin(y+z) - \\ - \cos(a)\,[1-\cos(y+z)] \end{matrix}\right\} dy\,dz -$$

$$-2(b-a)\int_a^b \int_a^b \sin\left(\frac{y+z}{2}\right)\cos\left(\frac{y-z}{2}\right) dy\,dz$$

$$= \int_a^b \int_a^b \left\{\begin{matrix} \cos(a+y+z) - \cos(b+y+z) + [\sin(b)-\sin(a)]\sin(y+z) + \\ +[\cos(b)-\cos(a)] + [\cos(a)-\cos(b)]\cos(y+z) \end{matrix}\right\} dy\,dz -$$

$$-2(b-a)\int_a^b \int_a^b \frac{e^{\frac{j(y+z)}{2}} - e^{-\frac{j(y+z)}{2}}}{j^2} \cdot \frac{e^{\frac{j(y+z)}{2}} + e^{\frac{-j(y+z)}{2}}}{2} dy\,dz$$

$$= \int_a^b \left\{\begin{matrix} \sin(a+y+z) - \sin(b+y+z) - [\sin(b)-\sin(a)]\cos(y+z) + \\ +[\cos(b)-\cos(a)]x + [\cos(a)-\cos(b)]\sin(y+z) \end{matrix}\right\}_a^b dz -$$

$$-(b-a)\int_a^b \int_a^b \frac{e^{jy} + e^{jz} - e^{-jz} - e^{-jy}}{j^2} dx\,dz$$

$$= \int_a^b \left\{\begin{matrix} \sin(a+b+c) - \sin(2a+z) - \sin(2b+z) + \sin(a+b+z) - \\ -[\sin(b)-\sin(a)][\cos(b+z)-\cos(a+z)] + \\ +[\cos(b)-\cos(a)](b-a) \end{matrix}\right\} dz +$$

$$+[\cos(a)-\cos(b)]\int_a^b [\sin(b+z)-\sin(a+z)]\,dz - (b-a)\int_a^b \int_a^b [\sin(y)+\sin(z)]\,dydz$$

$$= \int_a^b \left\{\begin{matrix} \sin(a+b+z) - \sin(2a+z) - \sin(2b+z) + \sin(a+b+z) - \\ -[\sin(b)-\sin(a)][\cos(b+z)-\cos(a+z)] + [\cos(b)-\cos(a)](b-a) \end{matrix}\right\} dz +$$

$$+[\cos(a)-\cos(b)]\int_a^b [\sin(b+z)-\sin(a+z)]\,dz - (b-a)\int_a^b \int_a^b [\sin(y)+\sin(z)]\,dz$$

$$\Omega(a,b)=\int_a^b\begin{Bmatrix}2\sin(a+b+z)-\sin(2a+z)-\sin(2b+z)-\\-[\sin(b)-\sin(a)][\cos(b+z)-\cos(a+z)]+\cos[(b)-\cos(a)]\,(b-a)\end{Bmatrix}dz+$$

$$+[\cos(a)-\cos(b)][\cos(a+z)-\cos(b+z)]_a^b-(b-a)\int_a^b[y\sin(z)-\cos(y)]_a^b\,dz$$

$$=\begin{Bmatrix}-2\cos(a+b+z)+\cos(2a+z)+\cos(2b+z)-[\sin(b)-\sin(a)]\cdot\\\cdot[\sin(b+z)-\sin(a+z)]+[\cos(b)-\cos(a)](b-a)z\end{Bmatrix}_a^b+$$

$$+[\cos(a)-\cos(b)][\cos(a+b)-\cos(2a)-\cos(2b)+\cos(a+b)]-$$

$$-(b-a)\int_a^b[(b-a)\sin(z)+\cos(a)-\cos(b)]\,dz$$

$$=-2\cos(a+2b)+2\cos(2a+b)+\cos(2a+b)\ -\cos(3a)+\cos(3b)-\cos(a+2b)-$$

$$-[\sin(b)-\sin(a)][\sin(2b)-\sin(a+b)-\sin(a+b)+\sin(2a)]=$$

$$=3\cos(2a+b)-3\cos(a+2b)-\cos(3a)+\cos(3b)-$$

$$-[\sin(b)-\sin(a)][\sin(2a)+\sin(2b)-2\sin(a+b)]$$

$$+[\cos(a)-\cos(b)][2\cos(a+b)-\cos(2a)-\cos(2b)-(b-a)^2]-$$

$$-(b-a)\{[\cos(a)-\cos(b)](b-a)-(b-a)[\cos(b)-\cos(a)]\}$$

$$=3[\cos(2a+b)-\cos(a+2b)]-\cos(3a)+\cos(3b)-$$

$$-[\sin(b)-\sin(a)][\sin(2a)-2\sin(a+b)+\sin(2b)]+$$

$$+[\cos(a)-\cos(b)][2\cos(a+b)-\cos(2a)-\cos(2b)-3(b-a)^2];\ 0<a\le b<2\pi$$

721. Find the shortest way to prove that:

$$\int_0^1\tan^{-1}\left(\sqrt{x}\right)^2\log(x)\,dx=-\pi G+\frac{7\zeta(3)}{4}+\frac{3\pi}{2}-\frac{5\pi^2}{24}-3\log(2)$$

where G is the Catalan Constant

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Tobi Joshua-Nigeria

$$I=\int_0^1\tan^{-1}\left(\sqrt{x}\right)^2\log x\,dx$$

$$I=4\int_0^1 \tan^{-1}(x)^2\log x\,dx$$

$$I=4\left[\left(\frac{x^2\log x}{2}-\frac{x^2}{4}\right)\tan^{-1}x_0^1-2\int_0^1\frac{\tan^{-1}x}{1+x^2}\left(\frac{x^2\log x}{2}-\frac{x^2}{4}\right)dx\right]$$

$$I=4\left[\frac{-\pi^2}{64}-\int_0^1\frac{x^2\log x\tan^{-1}x}{1+x^2}dx+\frac{1}{2}\int_0^1\frac{x^2\tan^{-1}x}{1+x^2}dx\right]$$

$$I=4\left[\frac{-\pi^2}{64}-\int_0^1\log x\tan^{-1}x\,dx+\int_0^1\frac{\log x\tan^{-1}x}{1+x^2}dx+\frac{1}{2}\int_0^1\tan^{-1}x\,dx-\frac{1}{2}\int_0^1\frac{\tan^{-1}x}{1+x^2}dx\right]$$

$$I=4\left[\frac{-\pi^2}{64}-\left(-\frac{\pi}{4}+\frac{\log 2}{2}-\int_0^1\frac{x\log x}{1+x^2}\right)+\left(\int_0^{\frac{\pi}{4}}x\log(\tan x)\,dx\right)+\frac{1}{2}\left(\frac{\pi}{4}-\frac{\log 2}{2}\right)-\frac{\pi^2}{64}\right]$$

$$I=4\left[\frac{-\pi^2}{32}+\frac{3\pi}{8}-\frac{3\log 2}{4}+\left(\int_0^1 x^{2k+1}\log x\,dx\sum_{k=0}^{\infty}(-1)^k\right)-2\left(\int_0^{\frac{\pi}{4}}x\sum_{k=0}^{\infty}\frac{\cos(2(2k+1)x)}{(2k+1)}dx\right)\right]$$

$$I=4\left[\frac{-\pi^2}{32}+\frac{3\pi}{8}-\frac{3\log 2}{4}-\frac{1}{4}\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^2}\right)-\frac{1}{2}\left(\frac{\pi}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}-\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}\right)\right]$$

$$I=4\left[\frac{-\pi^2}{32}+\frac{3\pi}{8}-\frac{3\log 2}{4}-\frac{1}{4}\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^2}\right)-\frac{1}{2}\left(\frac{\pi}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\right)-\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}\right]$$

$$I=4\left[\frac{-5\pi^2}{96}+\frac{3\pi}{8}-\frac{3\log 2}{4}-\frac{1}{2}\left(\frac{\pi}{2}G-\frac{7\zeta(3)}{8}\right)\right]$$

$$I=\left[\frac{-5\pi^2}{24}+\frac{3\pi}{2}-3\log 2-\pi G+\frac{7\zeta(3)}{4}\right]$$

722. $\Omega(a,b)=\int_0^{\pi}\left(\frac{x\cdot\sin x}{a+b\cos x}\right)dx,\ a>b,\ b\in(-1,1)$. **Find:**

$$\Psi=\int_0^1\left(\frac{b\cdot\Omega(1,b)}{\sqrt{1-b^2}}\right)db$$

Proposed by Abdul Hafeez Ayinde-Nigeria

Solution 1 by Mokhtar Khassani-Mostaganem-Algerie

$$\Omega(a,b)=\int_0^{\pi}\frac{x\sin x}{a+b\cos x}dx=-\frac{1}{b}\int_0^{\pi}\frac{x\cdot\frac{\partial}{\partial x}(a+b\cos x)}{a+b\cos x}dx$$

$$=-\frac{1}{b}[x\ln(a+b\cos x]_0^x+\frac{1}{b}\int_0^{\pi}\ln(a+b\cos x)\,dx$$

$$=-\frac{\pi\ln(a-b)}{b}+\frac{2}{b}\int_0^{\frac{\pi}{2}}\ln(a+b\cos 2x)\,dx$$

$$=-\frac{\pi\ln(a-b)}{b}+\frac{2}{b}\int_0^{\frac{\pi}{2}}\ln\left\{\left(\sqrt{a+b}\right)^2\cos^2 x+\left(\sqrt{a-b}\right)^2\sin^2 x\right\}dx \text{ as } a>b$$

$$=-\frac{\pi\ln(a-b)}{b}+\frac{2}{b}\cdot\pi\ln\left\{\frac{\sqrt{a+b}+\sqrt{a-b}}{2}\right\}=\frac{\pi}{b}\ln\left\{\frac{2a+2\sqrt{a^2-b^2}}{4(a-b)}\right\}$$

$$=\frac{\pi}{b}\ln\left(\frac{a+\sqrt{a^2-b^2}}{2a-2b}\right),\text{ where } a>b$$

— — — — — — — — — — — — — — — — — —

$$\int_0^1\frac{b\Omega(1,b)}{\sqrt{1-b^2}}db=\int_0^1\frac{b}{\sqrt{1-b^2}}\cdot\frac{\pi}{b}\ln\left(\frac{\sqrt{1-b^2}+1}{2(1-b)}\right)db$$

$$=\pi\int_0^{\frac{\pi}{2}}\ln\left\{\frac{\cos\theta+1}{2(1-\sin\theta)}\right\}d\theta=\pi\int_0^{\frac{\pi}{2}}\ln\left\{\frac{2\cos^2\frac{\theta}{2}}{2\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)}\right\}d\theta$$

$$=2\pi\int_0^{\frac{\pi}{4}}\ln\left(\frac{\cos^2\theta}{2\sin^2\left(\frac{\pi}{4}-\theta\right)}\right)d\theta=4\pi\int_0^{\frac{\pi}{4}}\ln\cos\theta\,d\theta-4\pi\int_0^{\frac{\pi}{4}}\ln\left\{\sqrt{2}\sin\left(\frac{\pi}{4}-\theta\right)\right\}d\theta$$

$$=4\pi\int_0^{\frac{\pi}{4}}\ln\cos\theta\,d\theta-4\pi\int_0^{\frac{\pi}{4}}\ln\left(\sqrt{2}\sin\theta\right)d\theta$$

$$=4\pi\int_0^{\frac{\pi}{4}}\ln\cot\theta\,d\theta-4\pi\ln\sqrt{2}\cdot\frac{\pi}{4}=4\pi G-\frac{\pi^2\ln 2}{2}$$

Solution 2 by Kartick Chandra Betal-India

$$\Omega = (a,b) = \int_0^{\pi} \frac{x\sin x}{a+b\cos x}dx = \left\{-\frac{x\log(a+b\cos x)}{b}\right\}_0^{\pi} + \frac{1}{b}\int_0^{\pi}\log(a+b\cos x)\,dx =$$

$$= \frac{\pi\log\left(\frac{1}{a-b}\right)}{b} + \frac{1}{b}\left(\pi\log a + \int_0^{\frac{b}{a}}\int_0^{\pi}\frac{\cos x}{1+y\cos x}dx\,dy\right)$$

$$= \frac{\pi\log\left(\frac{a}{a-b}\right)}{b} + \frac{1}{b}\int_0^{\frac{b}{a}}\frac{1}{y}\int_0^{\pi}\left(1-\frac{1}{1+y\cos x}\right)dx\,dy =$$

$$= \frac{\pi\log\left(\frac{a}{a-b}\right)}{b} + \frac{1}{a}\int_0^{\frac{b}{a}}\left(\frac{\pi}{y}-\frac{2}{y}\int_0^{\frac{\pi}{2}}\frac{1}{1-y^2\cos^2 x}dx\right)dy$$

$$= \frac{\pi\log\left(\frac{a}{a-b}\right)}{b} + \frac{1}{b}\int_0^{\frac{b}{a}}\left(\frac{\pi}{y}-\frac{2}{y}\int_0^{+\infty}\frac{1}{1-y^2+x^2}dx\right)dy$$

$$= \frac{\pi\log\left(\frac{a}{a-b}\right)}{b} + \frac{\pi}{b}\int_0^{\frac{b}{a}}\left(\frac{1}{y}-\frac{1}{y\sqrt{1-y^2}}\right)dy = \frac{\pi\log\left(\frac{a+\sqrt{a^2-b^2}}{2a-2b}\right)}{b}$$

$$\therefore \int_0^1 \frac{b\Omega(1,b)}{\sqrt{1-b^2}}db = \pi\int_0^1\frac{\log\left(\sqrt{1-b^2}+1\right)-\log 2-\log(1-b)}{\sqrt{1-b^2}}db =$$

$$= \pi\int_0^{\frac{\pi}{2}}(\log(1+\cos x)-\log 2-\log(1-\sin x))\,dx$$

$$\int_0^{\frac{\pi}{2}}\left(2\log\left(\cos\left(\frac{x}{2}\right)\right)-\log\left(\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right)-\log(\cos x)\right)dx$$

$$= \pi\left\{\frac{\pi}{2}\log 2 + 6\int_0^{\frac{\pi}{4}} \log(\cos x)\,dx - 2\int_0^{\frac{\pi}{4}} \log(\sin x)\,dx\right\} = 4\pi G - \frac{\pi^2}{2}\log 2$$

723. Find:

$$\Omega = \int_0^1 (x^2 \cdot \log x \cdot \tan^{-1} x)\,dx$$

Proposed by Abdul Hafeez Ayinde-Nigeria

Solution 1 by Avishek Mitra-India

$$\Omega = \int_0^1 x^2 \log x \cdot \tan^{-1} x\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)} \int_0^1 x^{2n+1} \log x\,dx$$

$$= -\frac{1}{4}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(n+1)^2} \quad \left[\because I = \int_0^1 x^{2n+1}\log x\,dx = \frac{(-1)^1(1!)}{(2n+1+1)^{1+1}}\right]$$

$$= -\frac{1}{12}\left[\frac{4}{3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)} - \frac{2}{3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)} - \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^2}\right]$$

$$= -\frac{1}{12}\left[\frac{4}{3}\cdot\tan^{-1}(1) - \frac{2}{3}(1-\log 2) - (1-\eta(2))\right] = -\frac{1}{12}\left[\frac{4}{3}\cdot\frac{\pi}{4} - \frac{2}{3}(1-\log 2) - \left(1-\frac{\pi^2}{12}\right)\right]$$

$$= -\frac{1}{12}\left[\frac{\pi}{3} - \frac{2}{3}(1-\log 2) - \left(1-\frac{\pi^2}{12}\right)\right] \text{ (answer)}$$

Solution 2 by Kartick-Chandra Betal-India

$$\int_0^1 x^2 \ln x \tan^{-1} x\,dx = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\int_0^1 x^{2n-1+2}\ln x\,dx$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\int_0^1 x^{2n+1}\ln x\,dx = -\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)(2n+2)^2}$$

$$= -\sum_{n=1}^{\infty}\left[\frac{1}{3^2(2n-1)} - \frac{1}{3^2(2n+2)} - \frac{1}{3(2n+2)^2}\right](-1)^{n-1}$$

$$= -\frac{1}{9}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)} + \frac{1}{18}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)} + \frac{1}{12}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)^2}$$

$$= -\frac{1}{9}\cdot\frac{\pi}{4} - \frac{1}{18}\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n} - \frac{1}{12}\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n^2}$$

$$= -\frac{\pi}{36} - \frac{1}{18}\left[\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} - 1\right] - \frac{1}{12}\left[\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} - 1\right]$$

$$= -\frac{\pi}{36} - \frac{1}{18}(\ln 2 - 1) - \frac{1}{12}\left\{\frac{\zeta(2)}{2} - 1\right\}$$

$$= \frac{1}{12} + \frac{1}{18} - \frac{\pi}{36} - \frac{\ln 2}{18} - \frac{\pi^2}{144} = \frac{5}{36} - \frac{\pi}{36} - \frac{\ln 2}{18} - \frac{\pi^2}{144}$$

$$= \frac{(20 - 20\pi - 40\ln 2 - \pi^2)}{144}$$

Solution 3 by Abdul Mukhtar-Nigeria

$$\Omega = \int_0^1 (x^2\cdot\ln x\cdot\tan^{-1}x)\,dx;\ \Omega = \frac{1}{3}\int_0^1 \ln x\cdot\arctan x\,d(x^3)$$

$$\Omega = -\frac{1}{3}\int_0^1 x^2\arctan x\,dx - \frac{1}{3}\int_0^1\frac{x^3}{1+x^2}\ln x\,dx;\ \Omega = -\frac{1}{3}(I_1 + I_2)$$

$$I_1 = \int_0^1 x^2\arctan x\,dx$$

set $u = \arctan x$; $dv = x^2$; $du = \frac{1}{1+x^2}$; $v = \frac{1}{3}x^3$ *using* $ub - \int v\,du$

$$I_1 = \left[\frac{1}{3}x^3\arctan x\right]_0^1 - \frac{1}{3}\int_0^1\frac{x^3}{1+x^2}dx$$

$$I_1 = \left[\frac{1}{3}\left(\frac{\pi}{4}\right) - 0\right] - \frac{1}{3}\left(\int_0^1 x\,dx + \int_0^1\frac{x}{1+x^2}dx\right)$$

$$I_1 = \frac{\pi}{12} - \frac{1}{3}\left(\left[\frac{1}{2}x^2\right]_0^1 + \left[\frac{1}{2}\ln(1+x^2)\right]_0^1\right)$$

$$I_1 = \frac{\pi}{12} - \frac{1}{6} + \frac{1}{6}\ln(2) \therefore I_1 = \frac{\pi}{12} - \frac{1}{6} + \frac{1}{6}\ln(2)$$

For $I_2 = \int_0^1 \frac{x^3}{1+x^2}\ln x\, dx$

$$I_2 = \int_0^1 x\ln x\, dx - \int_0^1 \frac{x}{1+x^2}\ln x\, dx;\ I_2 = \left[\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right]_0^1 - \int_0^1 \frac{x\ln x}{1+x^2}dx$$

$$I_2 = -\frac{1}{4} - \int_0^1 \frac{x\ln x}{1+x^2}dx;\ I_2 = -\frac{1}{4} - H;\ H = \int_0^1 \frac{x\ln x}{1+x^2}dx$$

set $x^2 = t \therefore H = \frac{1}{4}\int_0^1 \frac{\ln t}{1+t}dt;\ H = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\int_0^1 t^n\ln t\, dt;\ H = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\frac{1}{n^2}$

$$H = \frac{1}{4}\left(-\frac{1}{2}\zeta(2)\right);\ H = \frac{-1}{8}\zeta(2) \therefore I_2 = -\frac{1}{4} - \frac{1}{8}\zeta(2) \therefore \Omega = \frac{-1}{3}(I_1 + I_2)$$

$$\Omega = -\frac{1}{3}\left(\frac{\pi}{12} - \frac{1}{6} + \frac{1}{6}\ln(2) + \left(-\frac{1}{4} - \frac{1}{8}\zeta(2)\right)\right)$$

$$\Omega = -\frac{1}{3}\left(\frac{1}{6}\ln(2) - \frac{5}{12} + \frac{\pi}{12} + \frac{1}{8}\zeta(2)\right)$$

Where ζ ***is zeta function***

Solution 4 by Nelson Javier Villaherrera Lopez-El Salvador

$$\Omega = \int_0^1 x^2\ln(x)\tan^{-1}(x)\, dx = \left\{\left[\frac{x^3}{3}\ln(x) - \frac{x^3}{9}\right]\tan^{-1}(x) - \frac{1}{9}\int \frac{3x^3\ln(x) - x^3}{1+x^2}dx\right\}_0^1 =$$

$$= \frac{1}{9}\left\{[3x^3\ln(x) - x^3]\tan^{-1}(x) - \int \frac{x^3[3\ln(x) - 1]}{1+x^2}dx\right\}_0^1$$

$$= \frac{1}{9}\left\{-\frac{\pi}{4} - \int_0^1 \frac{x(x^2+1-1)[3\ln(x)-1]}{1+x^2}dx\right\} =$$

$$= -\frac{1}{9}\left\{\frac{\pi}{4} + \int_0^1 x[3\ln(x) - 1]\left(1 - \frac{1}{1+x^2}\right)dx\right\} =$$

$$= -\frac{1}{9}\left\{\frac{\pi}{4} + \int_0^1 [3x\ln(x) - x]dx - \int_0^1 \frac{3x\ln(x) - x}{1+x^2}dx\right\}$$

$$= -\frac{1}{9}\left[\frac{\pi}{4}-\frac{1}{2}+3\int_0^1 x\ln(x)-3\int_0^1\frac{x\ln(x)}{1+x^2}dx+\int_0^1\frac{x}{1+x^2}dx\right]=$$

$$= -\frac{1}{9}\left\{\frac{\pi}{4}-\frac{1}{2}+3\left[\frac{x^2}{2}\ln(x)-\frac{x^2}{4}\right]_0^1+3\int_0^1\frac{x[-\ln(x)]}{1+x^2}dx+\frac{1}{2}[\ln(x^2+1)]_0^1\right\}$$

$$= -\frac{1}{9}\left\{\frac{\pi}{4}-\frac{1}{2}+\frac{3}{2}\left[x^2\ln(x)-\frac{x^2}{2}\right]_0^1-3\int_\infty^0\frac{ye^{-y}}{1+e^{-2y}}e^{-y}dy+\frac{\ln(2)}{2}\right\}=$$

$$= -\frac{1}{18}\left[\frac{\pi}{2}-1+\ln(2)-\frac{3}{2}+6\int_0^\infty y\frac{e^{-2y}}{1+e^{-2y}}dy\right]$$

$$= -\frac{1}{18}\left[\frac{\pi}{2}+\ln(2)-\frac{5}{2}+6\int_0^\infty y\sum_{k=1}^\infty(-1)^{k-1}e^{-2ky}\,dy\right]=$$

$$= -\frac{1}{18}\left[\frac{\pi}{2}+\ln(2)-\frac{5}{2}+6\sum_{k=1}^\infty(-1)^{k-1}\int_0^\infty ye^{-2ky}\,dy\right]$$

$$= -\frac{1}{18}\left[\frac{\pi}{2}+\ln(2)-\frac{5}{2}+6\sum_{k=1}^\infty\frac{(-1)^{k-1}}{4k^2}\int_0^\infty 2kye^{-2ky}2k\,dy\right]=$$

$$= -\frac{1}{18}\left[\frac{\pi}{2}+\ln(2)-\frac{5}{2}+\frac{3}{2}\sum_{k=1}^\infty\frac{(-1)^{k-1}}{k^2}\int_0^\infty ze^{-z}dz\right]$$

$$= -\frac{1}{36}\left[\pi+2\ln(2)-5+3\Gamma(1+1)\sum_{k=1}^\infty\frac{(-1)^{k-1}}{k^2}\right]=$$

$$= -\frac{1}{36}\left[\pi+2\ln(2)-5+3\left(\sum_{k=1}^\infty\frac{1}{k^2}-\frac{1}{2}\sum_{k=1}^\infty\frac{1}{k^2}\right)\right]=-\frac{1}{36}\left[\pi+2\ln(2)-5+\frac{3}{2}\sum_{k=1}^\infty\frac{1}{k^2}\right]$$

$$= -\frac{1}{36}\left[\pi+2\ln(2)-5+\frac{3}{2}\zeta(2)\right]=-\frac{1}{36}\left[\pi+2\ln(2)-5+\frac{3}{2}\cdot\frac{\pi^2}{6}\right]=$$

$$= -\frac{1}{36}\left[\frac{\pi^2}{4}+\pi+2\ln(2)-5\right];\zeta(x)=\sum_{k=1}^\infty\frac{1}{k^x}$$

724. A Trigonometric integral with a beautiful closed – form:

Compute the integral:

$$\int_0^{\frac{\pi}{3}} \frac{1+\sin(x)}{1+\cos(x)} \cdot \frac{1-\sec(x)}{1-\csc(x)} \tan^2(x)\, dx$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Mokhtar Khassani-Mostaganem-Algerie

$$\Omega = \int_0^{\frac{\pi}{3}} \frac{1+\sin x}{1+\cos x} \cdot \frac{1-\sec x}{1-\csc x} \tan^2 x\, dx = \int_0^{\frac{\pi}{3}} \frac{(1+\sin x)(1-\cos x)}{\sin^2 x} \cdot \frac{1-\cos x}{1-\sin x} \tan^3 x\, dx =$$

$$= \int_0^{\frac{\pi}{3}} \left(\frac{1-\cos x}{1-\sin x}\right)^2 \tan x\, dx = M + N - 2P$$

$$M = \int_0^{\frac{\pi}{3}} \tan x \frac{dx}{(1-\sin x)^2} = \int_0^{\frac{\pi}{3}} \left(\frac{1}{\cos x\,(1-\sin x)^2} - \frac{1}{\cos x\,(1-\sin x)}\right) dx =$$

$$= \int_0^{\frac{\pi}{3}} \left(\frac{1}{(1-\sin^2 x)(1-\sin x)^2} - \frac{1}{(1-\sin^2 x)(1-\sin x)}\right) d(\sin x)$$

$$= \frac{1}{8} \int_0^{\frac{\pi}{3}} \left(\frac{4}{(1-\sin x)^3} - \frac{2}{(1-\sin x)^2} - \frac{2}{1-\sin^2 x}\right) d(\sin x) =$$

$$= \frac{1}{4} \left\{\frac{\sin x}{(1-\sin x)^2} - \operatorname{argtanh}(\sin x)\right\}_0^{\frac{\pi}{3}} = 6 + \frac{7\sqrt{3}}{2} - \frac{\operatorname{argtanh}\left(\frac{\sqrt{3}}{2}\right)}{4}$$

$$N = \int_0^{\frac{\pi}{3}} \tan x \frac{\cos^2 x}{(1-\sin x)^2} dx = \int_0^{\frac{\pi}{3}} \frac{\sin x}{(1-\sin x)^2} d(\sin x) = \left\{\frac{1}{1-\sin x} + \log(1-\sin x)\right\}_0^{\frac{\pi}{3}} =$$

$$= 3 + 2\sqrt{3} + \log\left(1 - \frac{\sqrt{3}}{2}\right)$$

$$P=\int_0^{\frac{\pi}{3}}\tan x\frac{\cos x}{(1-\sin x)^2}dx=\int_0^{\frac{\pi}{3}}\left(\frac{\cos x}{(1-\sin x)^2}-\frac{1}{1-\sin x}\right)dx$$

$$=\int_0^{\frac{\pi}{3}}(1+\sin x)^2\sec^2 x\,dx-\{\tan x+\sec x\}_0^{\frac{\pi}{3}}=-1-\sqrt{3}+\left\{\tan x+\frac{2}{3}\tan^3 x+\frac{2}{3}\sec^3 x\right\}_0^{\frac{\pi}{3}}=$$

$$=\frac{11}{3}+2\sqrt{3}\ \therefore\ \Omega=\frac{5}{3}+\frac{3\sqrt{3}}{2}+\log\left(1-\frac{\sqrt{3}}{2}\right)-\frac{\operatorname{argtanh}\left(\frac{\sqrt{3}}{2}\right)}{4}$$

725.

$$\int_0^1\int_0^\infty t^2e^{-(t^2x^2+at)}\,dtdx=\frac{\sqrt{\pi}e^{\frac{a^2}{4}}}{4}\operatorname{erfc}\left(\frac{a}{2}\right)\cdot\left(\frac{2-a^2}{a^2}\right)+\frac{1}{2a}$$

Proposed by Abdul Hafeez Ayinde-Nigeria

Solution by Dawid Bialek-Poland

$$\int_0^1\int_0^\infty t^2e^{-(t^2x^2+at)}\,dt\,dx=\int_0^1\int_0^\infty t^2e^{-at}e^{-t^2x^2}dt\,dx\quad\textbf{(1)}$$

Integrating w.rot x, we get:

$$\int_0^1 t^2e^{-at}e^{-t^2x^2}\,dx\overset{u=tx}{=}te^{-at}\int_0^t e^{-u^2}du\overset{def.}{=}\frac{\sqrt{\pi}}{2}te^{-at}\operatorname{erf}(t)\quad\textbf{(2)}$$

Rewriting (1) with (2) and integrating w.r.t. t, we get:

$$\frac{\sqrt{\pi}}{2}\int_0^\infty te^{-at}\operatorname{erf}(t)\,dt\overset{I.B.P.}{=}\begin{vmatrix}f=\operatorname{erf}(t) & g'=te^{-at}\\ f'=\frac{2}{\sqrt{\pi}}e^{-t^2} & g=-\frac{t}{a}e^{-at}-\frac{1}{a^2}e^{-at}\end{vmatrix}=$$

$$=\frac{\sqrt{\pi}}{2}\left[e^{-at}\operatorname{erf}(t)\left(-\frac{t}{a}-\frac{1}{a^2}\right)\right]_0^\infty+\frac{1}{a}\underbrace{\int_0^\infty te^{-t^2-at}\,dt}_{I_1}+\frac{1}{a^2}\underbrace{\int_0^\infty e^{-t^2-at}\,dt}_{I_2}\quad\textbf{(3)}$$

$$I_1=\int_0^\infty te^{-t^2-at}\,dt\overset{t=u-\frac{a}{2}}{=}\int_{\frac{a}{2}}^\infty\left(u-\frac{a}{2}\right)e^{-u^2+\frac{a^2}{4}}du=e^{\frac{a^2}{4}}\int_{\frac{a}{2}}^\infty\left(u-\frac{a}{2}\right)e^{-u^2}du=$$

$$\overset{I.B.P.}{=}\left|\begin{array}{ll} f=u-\frac{a}{2} & g'=e^{-u^2} \\ f'=1 & g=-\frac{\sqrt{\pi}}{2}\mathrm{erfc}(u)\end{array}\right|=e^{\frac{a^2}{4}}\left[-\frac{\sqrt{\pi}}{2}\left(u-\frac{a}{2}\right)\mathrm{erfc}(u)\right]_{\frac{a}{2}}^{\infty}+\frac{\sqrt{\pi}}{2}e^{\frac{a^2}{4}}\int_{\frac{a}{2}}^{\infty}\mathrm{erfc}(u)\,du=$$

$$\overset{I.B.P.}{=}\frac{\sqrt{\pi}}{2}e^{\frac{a^2}{4}}\left[e\cdot\mathrm{erfc}(u)-\frac{e^{-u^2}}{\sqrt{\pi}}\right]_{\frac{a}{2}}^{\infty}=\frac{\sqrt{\pi}}{2}e^{\frac{a^2}{4}}\left[0-\frac{a}{2}\mathrm{erfc}\left(\frac{a}{2}\right)+\frac{1}{\sqrt{\pi}}e^{-\frac{a^2}{4}}\right]=$$

$$=\frac{1}{2}-\frac{\sqrt{\pi}a}{4}e^{\frac{a^2}{4}}\mathrm{erfc}\left(\frac{a}{2}\right)$$

$$I_2=\int_0^{\infty}e^{-t^2-at}\,dt\overset{t=u-\frac{a}{2}}{=}\int_{\frac{a}{2}}^{\infty}e^{-u^2+\frac{a^2}{4}}du=e^{\frac{a^2}{4}}\int_{\frac{a}{2}}^{\infty}e^{-u^2}\,du\overset{def}{=}\frac{\sqrt{\pi}}{2}e^{\frac{a^2}{4}}\mathrm{erfc}\left(\frac{a}{2}\right)$$

Rewriting (3) with calculated integrals I_1 and I_2, we get the final result:

$$\int_0^1\int_0^{\infty}t^2e^{-(t^2x^2+at)}\,dt\,dx=\frac{1}{a}I_1+\frac{1}{a^2}I_2=\frac{1}{a}\left[\frac{1}{2}-\frac{\sqrt{\pi}a}{4}e^{\frac{a^2}{4}}\mathrm{erfc}\left(\frac{a}{2}\right)\right]+$$

$$+\frac{1}{a^2}\cdot\frac{\sqrt{\pi}}{2}e^{\frac{a^2}{4}}\mathrm{erfc}\left(\frac{a}{2}\right)=\frac{1}{a}-\frac{\sqrt{\pi}}{4}e^{\frac{a^2}{4}}\mathrm{erfc}\left(\frac{a}{2}\right)+\frac{\sqrt{\pi}}{2a^2}e^{\frac{a^2}{4}}\mathrm{erfc}\left(\frac{a}{2}\right)=$$

$$=\frac{\sqrt{\pi}}{4}e^{\frac{a^2}{4}}\mathrm{erfc}\left(\frac{a}{2}\right)\left[\frac{2}{a^2}-1\right]+\frac{1}{2a}$$

726. Find:

$$\Omega=\int_0^1\left(\sum_{n=1}^{\infty}\left(\frac{(-1)^nx^{2n+1}H_{2n}}{2n+1}\right)\right)dx$$

Proposed by Mohamed Arahman Jama-Somalia

Solution 1 by Kartick Chandra Betal-India

$$\int_0^1\sum_{n=1}^{\infty}\frac{(-1)^nH_{2n}}{2n+1}x^{2n+1}dx=\sum_{n=1}^{\infty}(-1)^n\frac{H_{2n}}{(2n+1)(2n+2)}$$

$$=\sum_{n=1}^{\infty}(-1)^n\left\{\frac{1}{2n+1}-\frac{1}{2n+2}\right\}H_{2n}=\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}H_{2n}-\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+2}H_{2n}$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}\left(H_{2n+1}-\frac{1}{2n+1}\right)-\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+2}\left\{H_{2n+2}-\frac{1}{2n+2}-\frac{1}{2n+1}\right\}$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}H_{2n+1}-\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)^2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+2}H_{2n+2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+2)^2}+$$

$$+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)(2n+2)}$$

$$=\int_0^1 x^2(1-x)\ln(1-x)\sum_{n=1}^{\infty}(-1)^{n-1}x^{2n-2}-\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}+\frac{1}{4}\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n^2}+$$

$$+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)}-\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+2)}$$

$$=\int_0^1\frac{x^2(1-x)\ln(1-x)}{1+x^2}dx-(G-1)+\frac{1}{4}\left(\frac{\zeta(2)}{2}-1\right)+\left(\frac{\pi}{4}-1\right)-\frac{1}{2}(\ln 2-1)$$

$$=\int_0^1(1-x)\ln(1-x)\,dx-\int_0^1\frac{(1-x)\ln(1-x)}{1+x^2}dx-G+1+\frac{\pi^2}{48}-\frac{1}{4}+\frac{\pi}{4}-1-\frac{\ln 2}{2}+\frac{1}{2}$$

$$=\int_0^1 x\ln x\,dx-\int_0^1\frac{\ln(1-x)}{1+x^2}dx+\int_0^1\frac{x\ln(1-x)}{1+x^2}dx+\frac{\pi^2}{48}-G+\frac{\pi}{4}-\frac{\ln 2}{2}+\frac{1}{4}$$

$$=\left[\frac{x^2}{2}\ln x-\frac{x^2}{4}\right]_0^1-\left\{\frac{\pi}{8}\ln 2-G\right\}+\left\{\frac{\ln^2 2}{8}-\frac{5\pi^2}{96}\right\}+\frac{\pi^2}{48}-G+\frac{\pi}{4}-\frac{\ln 2}{2}+\frac{1}{4}$$

$$=-\frac{1}{4}-\frac{\pi}{8}\ln 2+G+\frac{\ln^2 2}{8}-\frac{5\pi^2}{96}+\frac{\pi^2}{48}-G+\frac{\pi}{4}-\frac{\ln 2}{2}+\frac{1}{4}$$

$$=-\frac{\ln 2}{2}-\frac{\pi}{8}\ln 2+\frac{\ln^2 2}{8}-\frac{\pi^2}{32}+\frac{\pi}{4}=\frac{\ln^2 2}{8}-\frac{\pi}{8}\ln 2-\frac{\ln 2}{2}+\frac{\pi}{4}-\frac{\pi^2}{32}$$

Solution 2 by Abdul Mukhtar-Nigeria

Find $\Omega=\int_0^1\left(\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n+1}H_{2n}}{2n+1}\right)dx$. ***Let*** $\Omega=\int_0^1 S(\Omega)\,dx$

$S(\Omega)=\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n+1}H_{2n}}{2n+1}$. ***We know*** $H_n=\int_0^1\frac{1-t^n}{1-t}dx$. ***So,*** $H_{2n}=\int_0^1\frac{1-t^{2n}}{1-t}dx$

$$S(\Omega) = (-1)^{n-1}x^{2n}\int_0^1 \frac{1-t^{2n}}{1-t}dx;\ S(\Omega) = \int_0^1 \left[\frac{(tx)^2}{1+(tx)^2} - \frac{x^2}{1+x^2}\right]\frac{dt}{1-t}$$

$$S(\Omega) = \frac{x^2}{1+x^2}\left(\int_0^1 \frac{dt}{1+t^2x^2} + \int_0^1 \frac{t\,dt}{1+t^2x^2}\right)$$

$$S(\Omega) = \frac{1}{2}\left[\frac{2x}{1+x^2}\left(\arctan(x) + \frac{\ln(1+x^2)}{1+x^2}\right)\right]$$

$$\therefore S(\Omega) = \frac{1}{2}(\arctan(x)\ln(1+x^2)) \therefore \Omega = \int_0^1 S(\Omega)\,dx$$

but $S(\Omega) = \frac{1}{2}(\arctan(x)\ln(1+x^2))$

$$\Omega = \int_0^1 \frac{1}{2}(\arctan(x)\ln(1+x^2))\,dx$$

$$\Omega = \frac{1}{2}\int_0^1 \arctan(x)\ln(1+x^2)\,dx;\ \Omega = \frac{1}{2}I$$

Now, $I = \int_0^1 \arctan(x)\ln(1+x^2)\,dx;\ \Omega = -\frac{1}{2}I$

$$I = \int_0^1 \arctan(x)\ln(1+x^2)\,dx$$

Set $u = \ln(1+x^2) \Leftrightarrow du = \frac{2x}{1+x^2}$

$$dv = \arctan(x) \Leftrightarrow v = x\arctan(x) - \frac{1}{2}\ln(1+x^2)$$

Using $uv - \int v\,du$

$$I = \left[\ln(1+x^2)\,x\arctan(x) - \frac{1}{2}\ln(1+x^2)\right]_0^1 - \int_0^1 x\arctan(x) - \frac{1}{2}\ln(1+x^2)\cdot\frac{2x}{1+x^2}dx$$

$$I = \left[\ln(2)\left(\frac{\pi}{4} - \frac{1}{2}\ln(2)\right)\right] - \int_0^1 \frac{2x^2}{1+x^2}\arctan(x) + \int_0^1 \frac{x}{1+x^2}\ln(1+x^2)$$

$$I = \left[\ln(2)\left(\frac{\pi}{4} - \frac{1}{2}\ln(2)\right)\right] - H + J$$

$$H=\int_0^1 \frac{2x^2}{1+x^2}\arctan x\,dx=2\int_0^1 \frac{x^2+1-1}{1+x^2}\arctan x\,dx$$

$$H=2\int_0^1 \arctan x\,dx-2\int_0^1 \frac{\arctan x}{1+x^2}dx$$

$$H=2\left[\left[x\arctan(x)-\frac{1}{2}\ln(1+x^2)-\frac{1}{2}\arctan^2(x)\right]\right]_0^1$$

$$H=2\left[\left[\frac{\pi}{4}-\frac{1}{2}\ln(2)-\left[\frac{1}{2}\left(\frac{\pi}{4}\right)^2\right]\right]\right]$$

$$H=\frac{\pi}{2}-\ln(2)-\frac{\pi^2}{16}$$

$$J=\int_0^1 \frac{x}{1+x^2}\ln(1+x^2)\,dx=\frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\ln(1+x^2)\,dx$$

Set $t=\ln(1+x^2)\Leftrightarrow dt=\frac{2x}{1+x^2}dx$

$$J=\frac{1}{2}\int_0^1 t\,dt=\frac{1}{2}\left[\frac{1}{2}t^2\right]_0^1=\frac{1}{2}\left[\frac{1}{2}\ln^2(1+x^2)\right]_0^1$$

$$J=\frac{1}{4}\ln^2(2)$$

$$I=\left[\ln(2)\left(\frac{\pi}{4}-\frac{1}{2}\ln(2)\right)\right]-H+J$$

$$I=\left[\ln(2)\left(\frac{\pi}{4}-\frac{1}{2}\ln(2)\right)\right]-\left(\frac{\pi}{2}-\ln(2)-\frac{\pi^2}{16}\right)+\frac{1}{4}\ln^2(2)$$

$$I=\frac{\pi}{4}\ln(2)-\frac{1}{2}\ln^2(2)+\frac{1}{4}\ln^2(2)-\frac{\pi}{2}+\ln(2)+\frac{\pi^2}{16}$$

$$I=\frac{\pi}{4}\ln(2)-\frac{1}{4}\ln^2(2)-\frac{\pi}{2}+\ln(2)+\frac{\pi^2}{16}$$

Now, $\Omega=-\frac{1}{2}I$

$$\Omega=-\frac{1}{2}\left[\frac{\pi}{4}\ln(2)-\frac{1}{4}\ln^2(2)-\frac{\pi}{2}+\ln(2)+\frac{\pi^2}{16}\right]$$

727. Find the value of $\alpha > 2$ for which the area bounded by $y = \frac{1}{x}, y = \frac{1}{2x-1}$

$x = 2, x = \alpha$ is $\log\left(\frac{4}{\sqrt{5}}\right)$

Proposed by Sridhar Rao – India

Solution 1 by Daniel Sitaru – Romania

$$\Gamma_1: y = \frac{1}{x}, \Gamma_2: y = \frac{1}{2x-1}, \Gamma_1 \cap \Gamma_2 = \{(1,1)\}$$

$$x \ge 1 \to 2x - 1 \ge x \to \frac{1}{x} \ge \frac{1}{2x-1}$$

$$Area = \int_2^\alpha \left(\frac{1}{x} - \frac{1}{2x-1}\right) dx = \log\left(\frac{\alpha}{2}\right) - \log\sqrt{\frac{2\alpha - 1}{2}}$$

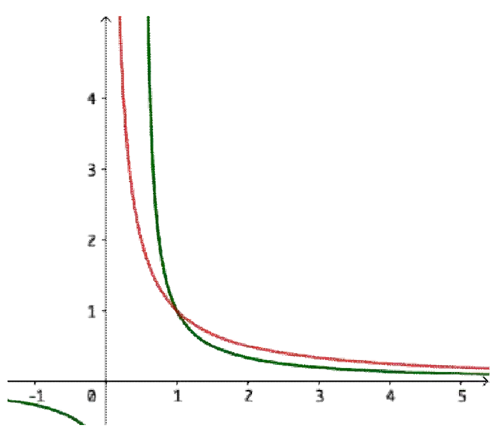
$$\log\left(\frac{\alpha}{2}\right) - \log\sqrt{\frac{2\alpha-1}{2}} = \log\left(\frac{4}{\sqrt{5}}\right)$$

$$\frac{\alpha}{2\sqrt{\frac{2\alpha-1}{2}}} = \frac{4}{\sqrt{5}}, \qquad 5\alpha^2 = \frac{64(2\alpha-1)}{3}$$

$$15\alpha^2 - 128\alpha + 64 = 0, \alpha = \frac{128+112}{30} = 8$$

Solution 2 by Adrian Popa – Romania

$$y = \frac{1}{x}; y = \frac{1}{2x-1};\ x = 2; x = \alpha;\ A = \int_2^\alpha \left(\frac{1}{x} - \frac{1}{2x-1}\right) dx = \log\frac{4}{\sqrt{5}}$$



$$-\to y=\frac{1}{x};x>0$$

$$-\to y=\frac{1}{2x-1};x>0$$

$$A=\ln x\Big|_2^{\alpha}-\frac{1}{2}\ln\left(x-\frac{1}{2}\right)\Big|_2^{2}=\ln\frac{x}{\sqrt{\left(x-\frac{1}{2}\right)}}\Big|_2^{\alpha}=\ln\frac{\alpha}{\sqrt{\left(\alpha-\frac{1}{2}\right)}}-\ln\frac{2}{\sqrt{\frac{3}{2}}}=$$

$$=\ln\frac{\alpha}{\sqrt{\alpha-\frac{1}{2}}}\cdot\frac{\sqrt{3}}{2\sqrt{2}}=\ln\frac{4}{\sqrt{5}}$$

$$\frac{\sqrt{3}\alpha}{2\sqrt{2}\sqrt{\alpha-\frac{1}{2}}}=\frac{4}{\sqrt{5}}\Rightarrow\alpha\sqrt{15}=8\sqrt{2}\sqrt{\alpha-\frac{1}{2}}$$

$$\alpha\sqrt{15}=8\sqrt{2\alpha-1}\,|\uparrow^2\Rightarrow 15\alpha^2=64(2\alpha-1)$$

$$15\alpha^2-128\alpha+64=0$$

$$\Delta=128^2-4\cdot15\cdot64=16\cdot384-3840=12\cdot544$$

$$\alpha_{1.2}=\frac{128\pm112}{30}\Rightarrow\left.\begin{array}{l}\alpha_1=8>2\\ \alpha_2=\frac{16}{30}<2\end{array}\right\}\Rightarrow\alpha=8$$

728.

$$\int_0^1\frac{\ln^2(1+x^2)-2\ln x\ln(1+x^2)}{1+x^2}dx$$

Proposed by Abdul Hafeez Ayinde-Nigeria

Solution 1 by Kartick Chandra Betal-India

$$\int_0^1\frac{\ln^2(1+x^2)-2\ln x\ln(1+x^2)}{1+x^2}dx$$

$$=\int_1^{\infty}\frac{\ln^2(1+x^2)+4\ln^2 x-4\ln x\ln(1+x^2)+2\ln x\{\ln(1+x^2)-2\ln x\}}{1+x^2}dx$$

$$= \int_1^{\infty} \frac{\ln^2(1+x^2) - 2\ln x \ln(1+x^2)}{1+x^2} dx$$

$$2I = \int_0^{\infty} \frac{\ln^2(1+x^2) - 2\ln x \ln(1+x^2)}{1+x^2} dx = \int_0^{\infty} \frac{\ln(1+x^2)\ln\left(1+\frac{1}{x^2}\right)}{1+x^2} dx$$

$$= \int_0^{\frac{\pi}{2}} \ln \sec^2 x \cdot \ln(\csc^2 x)\, dx = 4\int_0^{\frac{\pi}{2}} \ln(\sin x)\ln(\cos x)\, dx$$

$$I = 2\lim_{m,n\to 0}\left[\frac{\partial^2}{\partial m \partial n}\int_0^{\frac{\pi}{2}} (\sin x)^m (\cos x)^n dx\right] = \lim_{m,n\to 0}\left[\frac{\partial^2 \Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\partial m \partial n\, \Gamma\left(\frac{m+n+2}{2}\right)}\right]$$

$$= \frac{1}{2}\lim_{m\to 0}\left[\frac{\partial \Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\partial m\, \Gamma\left(\frac{m+2}{2}\right)}\left\{\psi\left(\frac{1}{2}\right) - \psi\left(\frac{m+2}{2}\right)\right\}\right]$$

$$= \frac{1}{4}\lim_{m\to 0}\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}\left[\left\{\psi\left(\frac{1}{2}\right) - \psi\left(\frac{m+2}{2}\right)\right\}\left\{\psi\left(\frac{m+1}{2}\right) - \psi\left(\frac{m+2}{2}\right)\right\} - \psi'\left(\frac{m+2}{2}\right)\right]$$

$$= \frac{1\Gamma^2\left(\frac{1}{2}\right)}{4\Gamma(1)}\left[\left\{\psi\left(\frac{1}{2}\right) - \psi(1)\right\}^2 - \psi'(1)\right] = \frac{\pi}{4}\left[4\ln^2 2 - \frac{\pi^2}{6}\right] = \pi \ln^2 2 - \frac{\pi^3}{24}$$

Solution 2 by Mokhtar Khassani-Mostaganem-Algerie

$$\Omega = \int_0^1 \frac{\log^2(1+x^2) - 2\log x \log(1+x^2)}{1+x^2} dx =$$

$$= \int_0^1 \left(\frac{(\log(1+x^2) - \log x)^2}{1+x^2} - \frac{\log^2 x}{1+x^2}\right) dx = M - N$$

$$N = \int_0^1 \frac{\log^2 x}{1+x^2} dx = \sum_{n=0}^{+\infty} (-1)^n \int_0^1 x^{2n}\log^2 x\, dx = 2\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^3} = 2\beta(3) = \frac{\pi^3}{16}$$

Note: $\therefore \underbrace{\beta(z) = \sum_{n=1}^{+\infty}\frac{(-1)^n}{(2n+1)^z}, z > 0}_{Dirichlet's\ beta\ function}$

$$M=\underbrace{\int_0^1\frac{(\log(1+x^2)-\log x)^2}{1+x^2}dx}_{x\to\tan y}=\int_0^{\frac{\pi}{4}}(-2\log(\cos x)-\log(\tan x))^2=$$

$$=\int_0^{\frac{\pi}{2}}\log^2\left(\frac{\sin(2x)}{2}\right)dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}\log^2\left(\frac{\sin x}{2}\right)dx$$

$$=\frac{\pi\log^2 2}{4}-\underbrace{\log 2\int_0^{\frac{\pi}{2}}\log(\sin x)\,dx}_{=-\frac{\pi}{2}\log 2}+\frac{1}{2}\int_0^{\frac{\pi}{2}}\log^2(\sin x)\,dx=$$

$$=\frac{3\pi}{4}\log^2 2+\frac{1}{16}\lim_{a\to\frac{1}{2}}\frac{\partial^2}{\partial a^2}\mathrm{B}\left(a,\frac{1}{2}\right)$$

$$=\frac{3\pi}{4}\log^2 2+\frac{1}{16}\lim_{a\to\frac{1}{2}}B\left(a,\frac{1}{2}\right)\left(\left(\Psi(a)-\Psi\left(a+\frac{1}{2}\right)\right)^2+\Psi_1(a)-\Psi_1\left(a+\frac{1}{2}\right)\right)=$$

$$=\pi\log^2 2+\frac{\pi^3}{48}\therefore\Omega=\pi\log^2 2-\frac{\pi^3}{24}$$

729. Find without softs:

$$\Omega=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-4x^2-9y^2}dxdy$$

Proposed by Jalil Hajimir-Canada

Solution 1 and generalization by Daniel Sitaru-Romania

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-4x^2-9y^2}\,dxdy=\left(\int_{-\infty}^{\infty}e^{-4x^2dx}\right)\left(\int_{-\infty}^{\infty}e^{-9y^2}dy\right)=$$

$$=2\left(\int_0^{\infty}e^{-4x^2dx}\right)\cdot 2\left(\int_0^{\infty}e^{-9y^2}dy\right)=2\cdot\frac{1}{2}\sqrt{\frac{\pi}{4}}\cdot 2\cdot\frac{1}{2}\sqrt{\frac{\pi}{9}}=\frac{\pi}{6}$$

GENERALIZATION:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} e^{-x_1^2-4x_2^2-9x_3^2-\dots-n^2x_n^2}\,dx_1dx_2\dots dx_n =$$

$$=\prod_{k=1}^{n}\left(\int_{-\infty}^{\infty} e^{-k^2x_k^2}dx_k\right)=2^n\prod_{k=1}^{n}\left(\int_{0}^{\infty} e^{-k^2x_k^2}dx_k\right)=2^n\prod_{k=1}^{n}\left(\frac{1}{2}\sqrt{\frac{\pi}{k^2}}\right)=\prod_{k=1}^{n}\left(\sqrt{\frac{\pi}{k^2}}\right)=\frac{(\sqrt{\pi})^n}{n!}$$

Solution 2 by Nelson Javier Villaherrera Lopez-El Salvador

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-4x^2-9y^2}\,dxdy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(2x)^2-(3y)^2}\,dxdy=$$

$$=\frac{1}{2}\cdot\frac{1}{3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x)^2-(y)^2}\,dxdy=\frac{1}{6}\int_{0}^{2\pi}\int_{0}^{\infty} e^{-\rho^2}d\rho d\varphi=\frac{-\pi}{6}\lim_{\rho\to\infty}\left(e^{-\rho^2}-1\right)=\frac{\pi}{6}$$

730. Find:

$$\Omega=\int_0^1\left(\frac{\tan^{-1}x\cdot\log^2(1+x^2)}{x^2}-\frac{\log^2(1+x^2)}{x(1+x^2)}\right)dx$$

Proposed by Abdul Hafeez Ayinde-Nigeria

Solution by Kartick Chandra Betal-India

$$\int_0^1\left[\frac{\tan^{-1}x\ln^2(1+x^2)}{x^2}-\frac{\ln^2(1+x^2)}{x(1+x^2)}\right]dx$$

$$=-\left[\frac{\ln^2(1+x^2)\tan^{-1}x}{x}\right]_0^1+\int_0^1\frac{1}{x}\left[\frac{\ln^2(1+x^2)}{1+x^2}+\frac{4x\ln(1+x^2)\tan^{-1}x}{1+x^2}\right]dx-$$

$$-\int_0^1\frac{\ln^2(1+x^2)}{x(1+x^2)}dx=-\frac{\ln^2 2\pi}{4}+4\int_0^1\frac{\ln(1+x^2)\tan^{-1}x}{1+x^2}dx$$

$$=-\frac{\pi\ln^2 2}{4}+4\int_0^{\frac{\pi}{4}}\frac{x\ln(1+\tan^2 x)}{1}dx=-\frac{\pi\ln^2 2}{4}-8\int_0^{\frac{\pi}{4}}x\ln\cos x\,dx$$

$$= -\frac{\pi\ln^2 2}{4} - 8\int_0^{\frac{\pi}{4}} x\left\{-\ln 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\cos(2nx)\right\}dx$$

$$= -\frac{\pi\ln^2 2}{4} + 8\ln 2\left[\frac{x^2}{2}\right]_0^{\frac{\pi}{4}} - 8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left[\frac{x\sin(2nx)}{2n} + \frac{\cos(2nx)}{(2n)^2}\right]_0^{\frac{\pi}{4}}$$

$$= -\frac{\pi\ln^2 2}{4} + \frac{\pi^2\ln 2}{4} - 8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left[\frac{\pi\sin\left(\frac{n\pi}{2}\right)}{8n} + \frac{\cos\left(\frac{n\pi}{2}\right) - 1}{4n^2}\right]$$

$$= -\frac{\pi\ln^2 2}{4} + \frac{\pi^2\ln 2}{4} - \pi\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin\left(\frac{n\pi}{2}\right)}{n^2} - 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos\left(\frac{n\pi}{2}\right)}{n^3} + 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}$$

$$= -\frac{\pi\ln^2 2}{4} + \frac{\pi^2\ln 2}{4} - \pi\sum_{n=1}^{\infty}\frac{\sin(2n-1)\frac{\pi}{2}}{(2n-1)^2} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{8n^3} + 2\cdot\frac{3}{4}\zeta(3)$$

$$= -\frac{\pi\ln^2 2}{4} + \frac{\pi^2\ln 2}{4} - \pi G - \frac{1}{4}\cdot\frac{3}{4}\zeta(3) + \frac{3}{2}\zeta(3) = -\pi G - \frac{3}{16}\zeta(3) + \frac{3}{2}\zeta(3)$$

$$= -\frac{\pi\ln^2 2}{4} + \frac{\pi^2\ln 2}{4} - \pi G + \frac{21}{16}\zeta(3) = \frac{\pi\ln 2}{4}(\pi - \ln 2) - \pi G + \frac{21}{16}\zeta(3)$$

731. Find:

$$\Omega = \int\frac{\sqrt{1+\sqrt[3]{x}}}{x}dx,\ x > 0$$

Proposed by Jalil Hajimir-Canada

Solution by Daniel Sitaru-Romania

$$x > 0, \int\frac{\sqrt{1+\sqrt[3]{x}}}{x}dx \overset{x=t^3}{\cong} \int\frac{\sqrt{1+t}\cdot 3t^2}{t^3}dt = 3\int\frac{\sqrt{1+t}}{t}dt =$$

$$= 3\int\frac{1+t}{t\sqrt{1+t}}dt = 3\int\frac{1}{t\sqrt{1+t}}dt + 3\int\frac{1}{\sqrt{1+t}}dt \overset{1+t=y^2}{\cong} 3\int\frac{1}{(y^2-1)y}\cdot 2ydy + 6\sqrt{1+t} =$$

$$= 6\int\frac{1}{y^2-1}dy + 6\sqrt{1+\sqrt[3]{x}} = 3log\left|\frac{y-1}{y+1}\right|dy + 6\sqrt{1+\sqrt[3]{x}} + C =$$

$$= 3log\left|\frac{\sqrt{1+t}-1}{\sqrt{1+t}+1}\right| + 6\sqrt{1+\sqrt[3]{x}} + C = 3log\left|\frac{\sqrt{1+\sqrt[3]{x}}-1}{\sqrt{1+\sqrt[3]{x}}+1}\right| + 6\sqrt{1+\sqrt[3]{x}} + C$$

732. **Find:**

$$\Omega = \int_0^1 (\{log(1+x)\}\cdot log\{1+x\})dx,$$

$$\{x\} = x - [x], [*] - great\ integer\ function$$

Proposed by Mokhtar Khassani-Mostaganem-Algerie

Solution 1 by Samir Haj Ali-Damascus-Syria

$$1+x=t, dx=dt, \Omega = \int_1^2 \{logt\}log\{t\} = \int_1^2 (logt-[logt])log(t-[t])dt =$$

$$\int_1^2 (logt)log(t-[t])dt - \int_1^2 ([logt])log(t-[t])dt =$$

$$= \int_1^2 logt\cdot log(t-[t])dt - \int_1^2 0\cdot log(t-[t])dt =$$

$$= \int_1^2 logt\cdot log(t-1)dt, \qquad t = 1-x, dt = -dx$$

$$\Omega = \int_{-1}^0 log(1-x)\cdot log(-x)dx = -\sum_{n=1}^{\infty}\frac{1}{n}x^n log(-x)dx$$

$$\Omega = \sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+1)^2} = \sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n} - \frac{(-1)^n}{n+1} - \frac{(-1)^n}{(n+1)^2}\right) =$$

$$= -log2 + (-log2+1) - \left(\frac{\pi^2}{12}-1\right) = 2(1-log2) - \frac{\pi^2}{12}$$

Solution 2 by Kartick Chandra Betal-India

$$\Omega = \int_0^1 (\{log(1+x)\}\cdot log\{1+x\})dx = \int_0^1 log(x+1)logxdx =$$

$$= (1\cdot log1 - 1)log(1+1) - \int_0^1 \frac{xlogx - x}{1+x}dx = -log2 - \int_0^1 \left(1-\frac{1}{1+x}\right)(logx-1)dx =$$

$$= -log2 - \int_0^1 (logx - 1)dx + \int_0^1 \frac{logx}{1+x}dx - \int_0^1 \frac{dx}{1+x} =$$

$$= -log2 - 1 \cdot log1 + 1 + 1 - \int_0^1 \frac{log(1+x)}{x}dx - log2 =$$

$$= -2log2 + 2 - \eta(2) = 2 - 2log2 - \frac{\pi^2}{12}$$

733. If

$$f(y) = \int_0^1 \frac{\sin^{-1}(xy)}{\sqrt{1-x^2}}dx$$

then show that

$$\int_0^1 f(y)\,dy = \frac{\pi^2}{8} - \log(2) ; \int_0^1 \frac{f(y)}{y}dy = \frac{7\zeta(3)}{8}$$

$$\int_0^1 \frac{f(y)}{\sqrt{y}}dy = \frac{\pi^2}{4} - \pi + 2\log(2)$$

$$\int_0^1 f(y)\log(y)\,dy = 2\log(2) - \frac{\pi^2}{6}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution 1 by Mokhtar Khassani-Mostaganem-Algerie

$$f(y) = \int_0^1 \frac{\arcsin(xy)}{\sqrt{1-x^2}}dx = \int_0^{\frac{\pi}{2}} \arcsin(y\sin x)\,dx = \sum_{n=0}^{\infty} \frac{y^{2n+1}\binom{2n}{n}}{(2n+1)4^n}\int_0^{\frac{\pi}{2}} \sin^{2n+1} x\,dx =$$

$$= \sum_{n=0}^{\infty} \frac{y^{2n+1}\binom{2n}{n}}{(2n+1)4^n} \cdot \frac{\sqrt{\pi}\Gamma(n+1)}{2\Gamma\left(n+1+\frac{1}{2}\right)} = \sum_{n=0}^{\infty} \frac{y^{2n+1}\binom{2n}{n}}{(2n+1)4^n} \cdot \frac{\sqrt{\pi}n!}{2\,\frac{(2n+2)!\sqrt{\pi}}{4^{n+1}(n+1)!}} =$$

$$=\sum_{n=0}^{\infty}\frac{y^{2n+1}}{(2n+1)^2}=\frac{Li_2(y)-Li_2(-y)}{2}=$$

$$M=\int_0^1 f(y)\,dy=\int_0^1\frac{Li_2(y)-Li_2(-y)}{2}dy=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\int_0^1 y^{2n+1}\,dy=$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2(n+1)}=\sum_{n=0}^{\infty}\left(\frac{1}{(2n+1)^2}+\frac{1}{2(n+1)}-\frac{1}{2n+1}\right)=\frac{\pi^8}{8}-\log 2$$

$$N=\int_0^1\frac{f(y)}{y}dy=\int_0^1\frac{Li_2(y)-Li_2(-y)}{2y}dy=\frac{1}{2}\{Li_3(x)-Li_3(-x)\}_0^1=$$

$$=\frac{1}{2}\left(\zeta(3)+\frac{3}{4}\zeta(3)\right)=\frac{7}{8}\zeta(3)$$

$$P=\int_0^1\frac{f(y)}{\sqrt{y}}dy=\int_0^1\frac{Li_2(y)-Li_2(-y)}{2\sqrt{y}}dy=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\int_0^1 y^{2n+\frac{1}{2}}dy=$$

$$=2\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2(4n+3)}=\sum_{n=0}^{\infty}\left(\frac{2}{(2n+1)^2}-\frac{1}{2\left(n+\frac{1}{2}\right)\left(n+\frac{3}{4}\right)}\right)=$$

$$=\frac{\pi^2}{4}-\frac{1}{2}\cdot\frac{\Psi\left(\frac{1}{2}\right)-\Psi\left(\frac{3}{4}\right)}{\frac{1}{2}-\frac{3}{4}}=\frac{\pi^2}{4}-\pi+2\log 2$$

$$\Omega=\int_0^1 f(y)\log y\,dy=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\int_0^1 y^{2n+1}\log y\,dy=$$

$$=-\sum_{n=0}^{\infty}\frac{1}{4(2n+1)^2(n+1)^2}=\sum_{n=0}^{\infty}\left(\frac{1}{2\left(n+\frac{1}{2}\right)(n+1)}-\frac{1}{(2n+1)^2}-\frac{1}{4(n+1)^2}\right)=$$

$$=\frac{1}{2}\cdot\frac{\Psi\left(\frac{1}{2}\right)-\Psi(1)}{\frac{1}{2}-1}-\frac{\pi^2}{8}-\frac{\pi^2}{24}=2\log 2-\frac{\pi^2}{6}$$

Solution 2 by Tobi Joshua-Nigeria

$$f(y)=\int_0^1\frac{\sin^{-1}(xy)}{\sqrt{1-x^2}}dx,\ x=\sin\theta,\ f(y)=\int_0^{\frac{\pi}{2}}\sin^{-1}(y\sin\theta)\,d\theta$$

$$f(y)=\int_0^{\frac{\pi}{2}}\int_0^y\frac{d\theta\,dz}{\sqrt{\left(\frac{1}{\sin\theta}\right)^2-z^2}},\ f(y)=\int_0^y\frac{dz}{z}\int_0^{\frac{\pi}{2}}\frac{\sin\theta d\theta}{\sqrt{\frac{1-z^2}{z^2}+(\cos\theta)^2}}$$

$$f(y)=-\int_0^y\frac{dz}{z}\int_0^{\frac{\pi}{2}}\frac{d(\cos\theta)}{\sqrt{\left(\frac{\sqrt{1-z^2}}{z}\right)^2+(\cos\theta)^2}},\ f(y)=\int_0^y\frac{dz}{z}\sinh^{-1}\left(\frac{z}{\sqrt{1-z^2}}\right)$$

$$f(y)=\int_0^y\frac{dz}{z}\log\left(\frac{z}{\sqrt{1-z^2}}+\sqrt{\frac{z^2}{1-z^2}+1}\right),\ f(y)=\int_0^y\frac{dz}{z}\log\left(\frac{z+1}{\sqrt{1-z^2}}\right)$$

$$f(y)=\frac{1}{2}\int_0^y\frac{dz}{z}\log\left(\frac{z+1}{1-z}\right),\ f(y)=\frac{1}{2}\int_0^y\frac{\log(1+z)-\log(1-z)}{z}dz$$

$$f(y)=\frac{1}{2}[Li_2(y)-Li_2(-y)],\ f(y)=\frac{1}{2}[Li_2(y)-Li_2(-y)]\quad (*)$$

$$(1)\ A=\int_0^1 f(y)\,dy$$

$$A=\frac{1}{2}[Li_2(y)-Li_2(-y)]dy$$

$$A=\frac{1}{2}\left[y\big(Li_2(y)-Li_2(-y)\big)_0^1+\int_0^1\ln(1-y)-\int_0^1\ln(1+y)\,dy\right]$$

$$A=\frac{1}{2}\left[\frac{\pi^2}{4}+\left(-\ln(1-y^2)+y\ln\left(\frac{1-y}{1+y}\right)\right)_0^1\right],\ A=\frac{1}{2}\left[\frac{\pi^2}{4}-2\ln(2)\right]$$

$$A=\left[\frac{\pi^2}{8}-\ln(2)\right],\ (2)$$

$$B=\int_0^1\frac{f(y)}{y}dy,\ B=\frac{1}{2}\int_0^1\frac{\big(Li_2(y)-Li_2(-y)\big)}{y}dy$$

$$B=\frac{1}{2}\left[\log(y)\big(Li_2(y)-Li_2(-y)\big)_0^1+\int_0^1\frac{\ln(y)}{y}\ln(1-y)-\int_0^1\frac{\ln(y)}{y}\ln(1+y)\,dy\right]$$

$$B=\frac{1}{2}\cdot\frac{\partial}{\partial a}\Big]_{a=0}\left[\int_0^1 y^{a-1}\ln(1-y)\,dy-\int_0^1 y^{a-1}\ln(1+y)\,dy\right]$$

$$B=\frac{1}{2}\cdot\frac{\partial}{\partial a}\Big]_{a=0}\sum_{k=0}^{\infty}\frac{1}{(k+1)}\left[-\int_0^1 y^a y^k dy-(-1)^k\int_0^1 y^a y^k\,dy\right]$$

$$B=\frac{1}{2}\cdot\frac{\partial}{\partial a}\Big]_{a=0}\sum_{k=0}^{\infty}\frac{1}{(k+1)}\left[-\frac{1}{(a+k+1)}-\frac{(-1)^k}{(a+k+1)}\right]$$

$$B=\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(k+1)}\left[\frac{1}{(k+1)^2}+\frac{(-1)^k}{(k+1)^2}\right],\ B=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{(k)}\left[\frac{1}{(k)^2}+\frac{(-1)^{k-1}}{(k)^2}\right]$$

$$B=\frac{1}{2}[\zeta(3)+\eta(3)],\ B=\frac{1}{2}\left[\zeta(3)+\frac{3}{4}\zeta(3)\right]$$

$$B=\left[\frac{7}{8}\zeta(3)\right]\ \otimes$$

(3) $C=\int_0^1\frac{f(y)}{\sqrt{y}}dy$

$$C=\frac{1}{2}\int_0^1\frac{\left(Li_2(y)-Li_2(-y)\right)}{\sqrt{y}}dy$$

$$C=\frac{1}{2}\left[2\sqrt{y}\left(\left(Li_2(y)-Li_2(-y)\right)_0^1+2\int_0^1\frac{1}{\sqrt{y}}\ln(1-y)-2\int_0^1\frac{1}{\sqrt{y}}\ln(1+y)\,dy\right)\right]$$

$$C=\frac{1}{2}\left[\frac{\pi^2}{2}+2\int_0^1\frac{1}{\sqrt{y}}\ln(1-y)-2\int_0^1\frac{1}{\sqrt{y}}\ln(1+y)\,dy\right]$$

$$C=\frac{\pi^2}{4}+\frac{1}{2}\left[2\int_0^1 y^{-\frac{1}{2}}\ln\left(\frac{1-y}{1+y}\right)dy\right],\ C=\frac{\pi^2}{4}+\frac{1}{2}\left[4\int_0^1\ln\left(\frac{1-y^2}{1+y^2}\right)dy\right]$$

$$C=\frac{\pi^2}{4}+2\left[y\ln\left(\frac{1-y^2}{1+y^2}\right)+2\int_0^1\frac{y^2dy}{1-y^2}+2\int_0^1\frac{y^2dy}{1+y^2}\right]$$

$$C=\frac{\pi^2}{4}+2\left[y\ln\left(\frac{1-y^2}{1+y^2}\right)+2\left(-y+\frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)\right)+2(y-\tan^{-1}y)\right]_0^1$$

$$C=\frac{\pi^2}{4}+2[(y-1)\ln(1-y)+(y+1)\ln(1+y)-y\ln(1+y^2)-2\tan^{-1}y]_0^1$$

$$C=\frac{\pi^2}{4}+2\left[(2)\ln(2)-\ln(2)-\frac{\pi}{2}\right],\ C=\frac{\pi^2}{4}+(2)\ln(2)-\pi\quad\otimes$$

(4) $D=\int_0^1 f(y)\ln y\,dy$

$$D=\frac{1}{2}\int_0^1\ln y\left(Li_2(y)-Li_2(-y)\right)dy,\ D=\frac{1}{2}\cdot\frac{\partial}{\partial s}\Big]_{s=0}\left[\int_0^1 y^2\left(Li_2(y)-Li_2(-y)\right)dy\right]$$

$$D=\frac{1}{2}\cdot\frac{\partial}{\partial s}\Big]_{s=0}\left[\frac{y^{s+1}}{s+1}\left(Li_2(y)-Li_2(-y)\right)_0^1+\int_0^1\frac{y^s}{s+1}\ln\left(\frac{1-y}{1+y}\right)dy\right]$$

$$D=\frac{1}{2}\cdot\frac{\partial}{\partial s}\bigg]_{s=0}\left[\frac{1}{s+1}\cdot\frac{\pi^2}{4}+\int_0^1\frac{y^s}{s+1}\ln(1-y)\,dy-\frac{1}{s+1}\int_0^1 y^s\ln(1+y)\,dy\right]$$

$$D=\frac{1}{2}\cdot\frac{\partial}{\partial s}\bigg]_{s=0}\left[\frac{\pi^2}{4(s+1)}+\sum_{k=0}^{\infty}\frac{1}{(k+1)}\left[-\frac{1}{s+1}\int_0^1 y^{s+k+1}\,dy-(-1)^k\int_0^1 y^{s+k+1}\,dy\right]\right]$$

$$D=\frac{1}{2}\cdot\frac{\partial}{\partial s}\bigg]_{s=0}\left[\frac{\pi^2}{4(s+1)}+\sum_{k=0}^{\infty}\frac{1}{(k+1)}\left[-\frac{1}{(s+1)(k+s+2)}-\frac{(-1)^k}{(s+1)(s+k+2)}\right]\right]$$

$$D=\left[-\frac{\pi^2}{8}+\sum_{k=0}^{\infty}\frac{1}{(k+1)}\left[\frac{1}{(k+2)}+\frac{1}{(k+2)^2}+\frac{(-1)^k}{(k+2)}+\frac{(-1)^k}{(k+2)^2}\right]\right]$$

$$D=\left[-\frac{\pi}{8}+\frac{1}{2}\left[(\psi(2)+\gamma)+\left(2-\frac{\pi^2}{6}\right)+(\ln 2)-(1-\ln 2)+\left(-2+\frac{\pi^2}{12}+2\ln 2\right)\right]\right]$$

$$D=\left[-\frac{\pi^2}{8}+\frac{1}{2}+\left(1-\frac{\pi^2}{12}\right)-\frac{1}{2}+\ln 2+\left(-1+\frac{\pi^2}{25}+\ln 2\right)\right]$$

$$D=\left[-\frac{\pi^2}{6}+2\ln 2\right]\ (*)$$

734. Find:

$$\int\left(x^2+x+1-\frac{8}{x^4}-\frac{4}{x^3}-\frac{2}{x^2}\right)6^{\left(x+\frac{2}{x}\right)}dx,\ x>0$$

Proposed by Jalil Hajimir-Canada

Solution by Remus Florin Stanca-Romania

$$\textit{Let } x+\frac{2}{x}=t\Rightarrow\left(1-\frac{2}{x^2}\right)dx=dt$$

$$I=\int\left(x^2\left(1-\frac{8}{x^6}\right)+x\left(1-\frac{4}{x^4}\right)+1-\frac{2}{x^2}\right)\cdot 6^{x+\frac{2}{x}}dx=$$

$$=\int\left(x^2\left(1-\frac{2}{x^2}\right)\left(1+\frac{4}{x^4}+\frac{2}{x^2}\right)+x\left(1-\frac{2}{x^2}\right)\left(1+\frac{2}{x^2}\right)+1-\frac{2}{x^2}\right)\cdot 6^{x+\frac{2}{x}}dx=$$

$$=\int\left(x^2+\frac{4}{x^2}+2+x+\frac{2}{x}+1\right)6^{x+\frac{2}{x}}\left(1-\frac{2}{x^2}\right)dx\Rightarrow I=\int\left(x^2+\frac{4}{x^2}+x+\frac{2}{x}+3\right)6^t\,dt;$$

$$x+\frac{2}{x}=t\Rightarrow x^2+\frac{4}{x^2}+4=t^2\Rightarrow$$

$$\Rightarrow x^2+\frac{4}{x^2}+3=t^2-1\Rightarrow I=\int(t^2-1+t)\,6^t dt=\int t^2 6^t\,dt-\int 6^t\,dt+\int t6^t\,dt \ \ \textit{(1)}$$

$$\int t^2 6^t\,dt=\frac{6^t}{\ln 6}t^2-\int\frac{6^t}{\ln 6}2t\,dt=\frac{6^t}{\ln 6}t^2-\frac{2}{\ln 6}\int 6^t\,t\,dt=$$

$$=\frac{6^t}{\ln 6}t^2-\frac{2}{\ln 6}\left(\frac{6^t}{\ln 6}t-\int\frac{6^t}{\ln 6}dt\right)=\frac{6^t}{\ln 6}t^2-\frac{2t6^t}{\ln^2 6}+\frac{2}{\ln^2 6}\cdot\frac{6^t}{\ln 6}+C \ \ \textit{(2)}$$

$$\int t6^t\,dt=\frac{6^t}{\ln 6}t-\int\frac{6^t}{\ln 6}dt=\frac{6^t}{\ln 6}-\frac{6^t}{\ln^2 6}+C \ \ \textit{(3)}$$

$$\overset{(1);(2);(3)}{\Rightarrow} I=6^t\cdot\frac{t^2\ln^2 6-2t\ln 6+2-\ln^2 6+t\ln^2 6-\ln 6}{\ln^3 6}+C=$$

$$=6^t\frac{(t\ln 6-1)^2+1-\ln^2 6+t\ln^2 6-\ln 6}{\ln^3 6}+C=$$

$$=6^{x+\frac{2}{x}}\cdot\frac{\left(\left(x+\frac{2}{x}\right)\ln 6-1\right)^2+1-\ln^2 6-\ln 6+\left(x+\frac{2}{x}\right)\ln^2 6}{\ln^3 6}+C=I$$

735. If for any complex number n, $Re(n)>0$,

$$\theta(n)=\int_{e^{-x}}^{\infty}e^{-nx^2}\,dx$$

then show that

$$\int_{-\infty}^{\infty}\theta(n)\,e^{-nx}dx=\frac{1}{2}n^{-\frac{n}{2}-\frac{3}{2}}\Gamma\left(\frac{n+1}{2}\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution 1 by Ekpo Samuel-Nigeria

Provided $Re(n)>0$

$$\theta(n)=\int_{e^{-x}}^{\infty}e^{-nx^2}\,dx=\int_{e^{-x}}^{\infty}e^{-(\sqrt{n}x)^2}\,dx$$

Let $u=\sqrt{n}x$

$$\therefore \theta(n) = \int_{\sqrt{n}e^{-x}}^{\infty} e^{-u^2}\,du = \frac{\sqrt{\pi}}{2\sqrt{n}}\mathrm{erfc}\left(\sqrt{n}e^{-x}\right)$$

Consider: $\int_{-\infty}^{\infty} e^{-nx}\frac{\sqrt{\pi}}{2\sqrt{n}}\mathrm{erfc}\left(\sqrt{n}e^{-x}\right)dx = \frac{\sqrt{\pi}}{2\sqrt{n}}\int_{-\infty}^{\infty} e^{-nx}\,\mathrm{erfc}\left(\sqrt{n}e^{-x}\right)dx$ ***(IBP)***

Let $u = \mathrm{erfc}\left(\sqrt{n}e^{-x}\right); du = \frac{2\sqrt{n}e^{-n(e^{-nx})}e^{-x}}{\sqrt{n}}; dv = e^{-nx}\,dx; v = -\frac{e^{-nx}}{n}$

$$\frac{\sqrt{\pi}}{2\sqrt{n}}\int_{-\infty}^{\infty} e^{-nx}\,\mathrm{erfc}\left(\sqrt{n}e^{-x}\right)dx = \frac{\sqrt{\pi}}{2\sqrt{n}}\int_{-\infty}^{\infty}\frac{e^{-nx}}{n}\left(\frac{2\sqrt{n}e^{-n(e^{-2x})}e^{-x}}{\sqrt{n}}\right)dx =$$

$$= \frac{1}{n}\int_{-\infty}^{\infty} e^{-n(n^{-2x})}\,e^{-x}e^{-nx}dx$$

$\frac{1}{n}\int_{-\infty}^{\infty}(e^{-x})^n\,e^{-n(e^{-x})^2}e^{-x}dx$***; let*** $u = e^{-x}du = -e^{-x}dx; \frac{1}{n}\int_0^{\infty} e^{-nu^2}u^2\,du$***; let*** $a = \sqrt{n}u$

$\frac{1}{n}\int_0^{\infty} e^{-a^2}\left(\frac{a}{\sqrt{n}}\right)^n\frac{da}{\sqrt{n}}$***; let*** $y = a^2 = \frac{1}{2n\left(\sqrt{n}\right)^{n+1}}\int_0^{\infty} e^{-y}\left(\sqrt{y}\right)^n\frac{dy}{\sqrt{y}} = \frac{1}{2}\left(n^{-}\left(\frac{n+1}{2}\right)-1\right)\Gamma\left(\frac{n+1}{2}\right)$

Simplifying: $= \frac{1}{2}n^{-\frac{n}{2}-\frac{3}{2}}\Gamma\left(\frac{n+1}{2}\right)$

Solution 2 by Ahmed Salama Hegazy-Cairo-Egypt

$$\theta(n) = \int_{e^{-x}}^{\infty} e^{-nx^2}\,dx = \int_0^{\infty} e^{-nx^2}\,dx - \int_0^{e^{-x}} e^{-nx^2}dx$$

$$\therefore \theta(n) = \frac{1}{\sqrt{n}}\int_0^{\infty} e^{-x^2}\,dx - \frac{1}{\sqrt{n}}\int_0^{\sqrt{n}e^{-x}} e^{-x^2}\,dx$$

$$\therefore \theta(n) = \frac{\sqrt{\pi}}{2\sqrt{n}} - \frac{\sqrt{\pi}}{2\sqrt{n}}\mathrm{erf}\left(\sqrt{n}e^{-x}\right) = \frac{\sqrt{\pi}}{2\sqrt{n}}\mathrm{erfc}\left(\sqrt{n}e^{-x}\right)$$

$$I = \int_{-\infty}^{\infty}\theta(n)\cdot e^{-nx}dx = \frac{\sqrt{\pi}}{2\sqrt{n}}\int_{-\infty}^{\infty}\mathrm{erfc}\left(\sqrt{n}e^{-x}\right)\cdot e^{-nx}dx$$

Let $y = \sqrt{n}e^{-x}, dy = -\sqrt{n}e^{-x}dx \therefore dx = -\frac{dy}{y}, y{:}\,0$ ***to*** ∞

$$\therefore I = \frac{\sqrt{\pi}}{2\sqrt{n}}\int_0^{\infty}\mathrm{erfc}(y)\cdot\left(\frac{y}{\sqrt{n}}\right)^n - \frac{dy}{y} = \frac{\sqrt{\pi}}{2\sqrt{n}(n)^{\frac{\pi}{2}}}\int_0^{\infty}\mathrm{erfc}(y)\,y^{n-1}dy$$

$$\therefore I=\frac{\sqrt{\pi}}{2\sqrt{n}\cdot n^{\frac{n}{2}}}\int_0^{\infty}\operatorname{erfc}(y)\cdot y^{n-1}dy$$

$$=\frac{\sqrt{\pi}}{2\sqrt{n}n^{\frac{n}{2}}}\left(y^n\operatorname{erfc}(y)-(n-1)\int_0^{\infty}\operatorname{erfc}(y)\,y^{n-1}\,dy+\frac{2}{\sqrt{\pi}}\int_0^{\infty}y^n\cdot e^{-y^2}\,dy\right)$$

$$\therefore I=\frac{\sqrt{\pi}}{2\sqrt{n}n^{\frac{n}{2}}}\cdot\frac{2}{n\sqrt{\pi}}\int_0^{\infty}y^ne^{-y^2}\,dy,\ \textit{let}\ y^2=u,\ dy=\frac{du}{2\sqrt{u}}$$

$$\therefore I=\frac{1}{\sqrt{n}\cdot n^{\frac{n}{2}}}\cdot\frac{1}{2}\int_0^{\infty}u^{\frac{n}{2}}\cdot e^{-u}\cdot\frac{du}{\sqrt{u}}=\frac{1}{2}(n)^{-\left(\frac{n}{2}+\frac{1}{2}\right)}\Gamma\left(\frac{n+1}{2}\right)$$

Solution 3 by Mokhtar Khassani-Mostaganem-Algerie

$$\theta(n)=\int_{e^{-x}}^{\infty}e^{-ny^2}\,dy=\frac{1}{\sqrt{n}}\int_{\sqrt{n}e^{-x}}^{\infty}e^{-ny^2}\,dy=\frac{\sqrt{\pi}}{2\sqrt{n}}\operatorname{erfc}\left(\sqrt{n}e^{-x}\right)$$

$$M=\int_{-\infty}^{+\infty}\theta(n)\,e^{-nx}dx=\frac{\sqrt{\pi}}{2\sqrt{n}}\overbrace{\int_{-\infty}^{+\infty}\operatorname{erfc}\left(\sqrt{n}e^{-x}\right)e^{-nx}\,dx}^{y=\sqrt{n}e^{-x}}=\frac{\sqrt{\pi}}{2n^{\frac{n+1}{2}}}\int_0^{\infty}x^{n-1}\operatorname{erfc}(x)\,dx$$

$$\overset{IBP}{=}\frac{\sqrt{\pi}}{2n^{\frac{n+1}{2}}}\left\{\frac{1}{n}\operatorname{erfc}(x)\,x^{n-2}\right\}_0^{\infty}+\frac{1}{n^{\frac{n+3}{2}}}\overbrace{\int_0^{\infty}x^ne^{-x^2}\,dx}^{x^2=y}=\frac{1}{2n^{\frac{n+3}{2}}}\int_0^{\infty}x^{\frac{n-1}{2}}e^{-x}dx=\frac{\Gamma\left(\frac{n+1}{2}\right)}{2n^{\frac{n+3}{2}}}$$

736. Find:

$$\Omega=\int_0^1\left(\sum_{n=1}^{\infty}\left(\frac{d^n(x^n\log x)}{dx^n}\cdot\frac{x^n}{n!\cdot n^2}\right)\right)dx$$

Proposed by Abdul Hafeez Ayinde-Nigeria

Solution by Avishek Mitra-West Bengal-India

$$\Omega=\int_0^1\sum_{n=1}^{\infty}\left(\frac{d^n}{dx^n}(x^n\log x)\cdot\frac{x^n}{n!\cdot n^2}\right)dx=\int_0^1\sum_{n=1}^{\infty}\frac{1}{n^2}\left(n!\,(\log x+H_n)\cdot\frac{x^n}{n!}\right)dx$$

$$=\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^1 x^n\log x\,dx+\sum_{n=1}^{\infty}\frac{H_n}{n^2}\int_0^1 x^n\,dx=\sum_{n=1}^{\infty}\frac{1}{n^2(n+1)^2}+\sum_{n=1}^{\infty}\frac{H_n}{n^2(n+1)}$$

$$=-\sum_{n=1}^{\infty}\frac{1}{n^2}+2\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)-\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}+\sum_{n=1}^{\infty}\frac{H_n}{n^2}-\sum_{n=1}^{\infty}\left(\frac{H_n}{n}-\frac{H_n}{n+1}\right)$$

$$=-\zeta(2)+2+1-\sum_{n=1}^{\infty}\frac{1}{n^2}+2\zeta(3)-\left(\sum_{n=1}^{\infty}\frac{H_n}{n}-\sum_{n=1}^{\infty}\frac{H_{n+1}}{n+1}+\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}\right)$$

$$=3-2\zeta(2)+2\zeta(3)-\left(1+\sum_{n=1}^{\infty}\frac{1}{n^2}-1\right)=3-3\cdot\frac{\pi^2}{6}+2\zeta(3)=3-\frac{\pi^2}{2}+2\zeta(3)$$

737. Find without softs:

$$\Omega(n)=\int_0^{\infty}\left(\frac{x^n}{e^{\sqrt[4]{x}}}\cdot\sin(\sqrt[4]{x})\right)dx,n\in\mathbb{N}$$

Proposed by Abdul Mukhtar-Nigeria

Solution 1 by Kartick Chandra Betal-India

$$\Omega(n)=\int_0^{\infty}\frac{x^n}{e^{\sqrt[4]{x}}}\sin(\sqrt[4]{x})\,dx=4\int_0^{\infty}x^3\cdot x^{4n}\cdot e^{-x}\sin x\,dx$$

$$=4\,Imag\int_0^{\infty}x^{4n+3}\,e^{-x(1-i)}dx=4\cdot Imag.\left\{\frac{\Gamma(4n+4)}{(1-i)^{4n+4}}\right\}$$

$$=4\,Imag.\left\{\frac{\Gamma(4n+4)}{(-2i)^{2n+2}}\right\}=4\cdot\frac{\Gamma(4n+4)}{2^{2n+2}}.Imag\left\{\frac{1}{(-1)^{n+1}}\right\}$$

$$=\frac{\Gamma(4n+4)}{2^{2n}}(-1)^{n+1}.Imag.\{1\}=0$$

Solution 2 by Mokhtar Khassani-Mostaganem-Algerie

$$\Omega(n)=\int_0^{+\infty}\frac{x^n}{e^{\sqrt[4]{x}}}\sin(\sqrt[4]{x})\,dx=Im\underbrace{\int_0^{+\infty}x^ne^{-(1-i)\sqrt[4]{x}}dx}_{x=y^4}=$$

$$= 3Im\int_0^{+\infty} x^{4n+3}e^{-(1-i)x}\,dx = 3\,Im\frac{\Gamma(4n+4)}{(1-i)^{4n+4}} = 0$$

Solution 3 by Precious Itsuokor-Nigeria

$$\int_0^\infty \frac{x^n}{e^{\sqrt[4]{x}}}\sin\left(\sqrt[4]{x}\right)dx \text{ put } \sqrt[4]{x} = u;\ x = u^4;\ dx = 4u^3du$$

$$\Rightarrow 4\int_0^\infty \frac{u^{4n}}{e^u}\sin(u)\,u^3du = 4\int_0^\infty u^{4n+3}\,e^{-u}\sin(u)\,du = 4.\,Im\left[\int_0^\infty u^{4n+3}e^{-u}e^{iu}\,du\right]$$

$$= 4\left(Im\left[\int_0^\infty u^{4n+3}e^{-1(1-i)u}du\right]\right)$$

$$\text{Put } (1-i)u = t;\ u = \frac{t}{1-i};\ du = \frac{dt}{1-i}$$

$$\Rightarrow 4\,Im\left[\frac{1}{(1-i)^{4n+4}}\int_0^\infty t^{4n+3}\,e^{-t}dt\right] = 4\,Im\left[\frac{\Gamma(4n+4)}{(1-i)^{4n}+4}\right]$$

$$\text{Since } (1-i)^4 = -4$$

$$= 4Im\left[\frac{\Gamma(4n+4)}{(-4)^{n+1}}\right] = \frac{4}{4^{n+1}}\Gamma(4n+4)Im\left[\frac{1}{(-1)^{n+1}}\right] = 4^{-n}\Gamma(4n+4)\cdot 0 = 0$$

Solution 4 by Ovwie Edafe-Nigeria

$$\Omega(n) = \int_0^\infty \frac{x^n\sin\sqrt[4]{x}}{e^{\sqrt[4]{x}}}dx, n \in \mathbb{N};\text{ set } u = \sqrt[4]{x};\ x = u^4;\ dx = 4u^3du$$

$$4\int_0^\infty u^{4n+3}\sin u\,e^{-u}du.\text{ We know } \sin u = \frac{e^{iu}-e^{-iu}}{2i}$$

$$\frac{2}{i}\left[\int_0^\infty u^{4n+3}e^{-(1-i)u}dx - \int_0^\infty u^{4n+3}e^{-(1+i)u}\,du\right]$$

$$\text{By Laplace transformation } L(u^a) = \frac{a!}{s^{a+1}} \text{ and } \frac{1}{i} = -i$$

$$-2i\left[\frac{(4n+3)!}{(1-i)^{4n+4}} - \frac{(4n+3)!}{(1+i)^{4n+4}}\right]$$

$$\text{We know } (1-i)(1+i) = 2$$

……..

$$-2i\left[\frac{(1+i)^{4n+4}(4n+3)!-\{(1-i)^{4n+4}(4n+3)!\}}{(16)^{n+1}}\right]$$

$$-2i\left[\frac{(-4)^{n+1}(4n+3)!-\{(-4)^{n+1}(4n+3)!\}}{(16)^{n+1}}\right]$$

$$-2i\left[\frac{0}{(16)^{n+1}}\right]$$

$$0$$

$$\Omega(n)=\int_0^\infty\frac{x^n\sin\sqrt[4]{x}}{e^{\sqrt[4]{x}}}dx, n\in\mathbb{N}=0$$

738. **Find:**

$$\Omega=\int_0^\infty\left(\frac{\sin x}{x}\right)dx-\int_0^1\left(\frac{\sec(\tan^{-1}x)}{1+x}\right)dx$$

Proposed by Precious Itsuokor-Nigeria

Solution 1 by Mokhtar Khassani-Mostaganem-Algerie

$$\Omega=\int_0^\infty\frac{\sin x}{x}dx-\int_0^1\frac{\sec(\arctan x)}{1+x}dx=\int_0^\infty\int_0^\infty\sin(x)\,e^{-xy}dy\,dx-\underbrace{\int_0^1\frac{\sqrt{1+x^2}}{1+x}dx}_{x=\sin y}=$$

$$=\int_0^\infty\frac{1}{1+x^2}dx-\int_0^{\operatorname{argsinh}(1)}\frac{\cosh^2 x}{1+\sinh x}dx=\frac{\pi}{2}-M$$

$$M=\underbrace{\int_0^{\operatorname{argsinh}(1)}\frac{\cosh^2(x)}{1+\sinh x}dx}_{x=\log y}=\frac{1}{2}\int_1^{1+\sqrt{2}}\frac{(1+x)^2}{x^2(x^2+x-1)}dx=\frac{1}{2}\int_1^{1+\sqrt{2}}\left(1-\frac{1}{x^2}-\frac{2}{x}+\frac{8}{x^2+2x-1}\right)dx=$$

$$=\frac{1}{2}\left(\sqrt{2}+\frac{1}{1+\sqrt{2}}-2\ln\left(1+\sqrt{2}\right)\right)-\left\{2\sqrt{2}\operatorname{argtanh}\left(\frac{1+x}{\sqrt{2}}\right)\right\}_1^{1+\sqrt{2}}$$

$$=-\left(1-\sqrt{2}+\ln\left(1+\sqrt{2}\right)+2\sqrt{2}\operatorname{argtanh}\left(1-\sqrt{2}\right)\right)$$

$$\therefore\Omega=\frac{\pi}{2}+1-\sqrt{2}+\ln\left(1+\sqrt{2}\right)+2\sqrt{2}\operatorname{argtanh}\left(1-\sqrt{2}\right)$$

Solution 2 by Nelson Javier Villaherrera Lopez-El Salvador

$$\Omega=\int_0^\infty \frac{\sin(x)}{x}dx-\int_0^1 \frac{\sec[\tan^{-1}(x)]}{1+x}dx=L\left\{\frac{\sin(x)}{x}\right\}\Bigg|_{i=0}-\int_0^1 \frac{\sqrt{\sec^2[\tan^{-1}(x)]}}{1+x}dx=$$

$$=\int_0^\infty L\{\sin(x)\}\,dr-\int_0^1 \frac{\sqrt{1+\tan^2[\tan^{-1}(x)]}}{1+x}dx$$

$$=\int_0^\infty \frac{dr}{1+r^2}-\int_0^1 \frac{\sqrt{1+x^2}}{1+x}dx=[\tan^{-1}(r)]_0^\infty-\int_0^1 \frac{1+x^2}{(1+x)\sqrt{1+x^2}}dx=$$

$$=\frac{\pi}{2}-\int_0^1 \frac{1+2x+x^2-2x}{(1+x)\sqrt{1+x^2}}dx=\frac{\pi}{2}-\int_0^1 \frac{(1+x)^2-2x}{(1+x)\sqrt{1+x^2}}dx$$

$$=\frac{\pi}{2}-\int_0^1\left[\frac{1+x}{\sqrt{1+x^2}}-2\frac{x+1-1}{(1+x)\sqrt{1+x^2}}\right]dx=$$

$$=\frac{\pi}{2}-\int_0^1\left[\frac{1+x}{\sqrt{1+x^2}}-\frac{2}{\sqrt{1+x^2}}+\frac{2}{(1+x)\sqrt{1+x^2}}\right]dx$$

$$=\frac{\pi}{2}-\int_0^1\left[\frac{x}{\sqrt{1+x^2}}-\frac{1}{\sqrt{1+x^2}}+\frac{2}{(1+x)\sqrt{1+x^2}}\right]dx$$

$$=\frac{\pi}{2}-\left[\sqrt{1+x^2}\right]_0^1+\int_0^{\frac{\pi}{4}}\left\{\frac{1}{\sqrt{1+\tan^2(\theta)}}-\frac{2}{[1+\tan(\theta)]\sqrt{1+\tan^2(\theta)}}\right\}\sec^2(\theta)\,d\theta=$$

$$=1+\frac{\pi}{2}-\sqrt{2}+\int_0^{\frac{\pi}{4}}\left[\sec(\theta)-\frac{2\sec(\theta)}{1+\tan(\theta)}\right]d\theta$$

$$=1+\frac{\pi}{2}-\sqrt{2}+\left[\ln|\sec(\theta)+\tan(\theta)|-2\int\frac{d\theta}{\cos(\theta)+\sin(\theta)}\right]_0^{\frac{\pi}{4}}=$$

$$=1+\frac{\pi}{2}-\sqrt{2}+\ln\left(\sqrt{2}+1\right)-\sqrt{2}\int_0^{\frac{\pi}{4}}\frac{d\theta}{\cos\frac{(\theta)}{\sqrt{2}}+\sin\frac{(\theta)}{\sqrt{2}}}$$

$$= 1 + \frac{\pi}{2} - \sqrt{2} + \ln(\sqrt{2}+1) - \sqrt{2}\int_0^{\frac{\pi}{4}} \frac{d\theta}{\cos\left(\theta - \frac{\pi}{4}\right)} =$$

$$= 1 + \frac{\pi}{2} - \sqrt{2} + \ln(\sqrt{2}+1) - \sqrt{2}\left[\ln\left|\sec\left(\theta - \frac{\pi}{4}\right) + \tan\left(\theta - \frac{\pi}{4}\right)\right|\right]_0^{\frac{\pi}{4}}$$

$$= 1 + \frac{\pi}{2} - \sqrt{2} + \ln(\sqrt{2}+1) + \sqrt{2}\ln(\sqrt{2}-1) =$$

$$= 1 + \frac{\pi}{2} - \sqrt{2} + \ln(\sqrt{2}+1) - \sqrt{2}\ln(\sqrt{2}+1) = 1 + \frac{\pi}{2} - \sqrt{2} - (\sqrt{2}-1)\ln(\sqrt{2}+1)$$

739. Find without softs:

$$\Omega = \int_0^{2\pi} \left(\frac{x + \tan(\sin x)}{2 + \cos x}\right) dx$$

Proposed by Florică Anastase-Romania

Solution 1 by Hemn Hsain-Iraq

$$I = \int_0^{2\pi} \frac{x + \tan(\sin x)}{2 + \cos x} dx, \qquad I = \int_0^{2\pi} \frac{(2\pi - x) + \tan(\sin(2\pi - x))}{2 + \cos(2\pi - x)} dx$$

$$I = \int_0^{2\pi} \frac{(2\pi - x) - \tan(\sin x)}{2 + \cos x} dx, \qquad 2I = \int_0^{2\pi} \frac{2\pi}{2 + \cos x} dx$$

$$I = \int_0^{2\pi} \frac{\pi}{2 + \cos x} dx \Rightarrow d\theta = \frac{dz}{iz}, \cos x = \frac{z + z^{-1}}{2}$$

$$I = \oint \frac{\frac{\pi dz}{iz}}{2 + \frac{z^2+1}{2z}} = \frac{2\pi}{i}\oint \frac{dz}{z^2 + uz + 1}, \qquad I = 2\pi i\, Res\left(\frac{1}{z^2 + uz + 1}, -2 + \sqrt{3}\right)$$

$$I = \frac{2\pi}{i}\left(2\pi i\left(\frac{1}{2\sqrt{3}}\right)\right) = \frac{2\pi^2}{\sqrt{3}}$$

Solution 2 by Samir HajAli-Damascus-Syria

$$\Omega = \int_0^{2\pi} \frac{x + \tan(\sin x)}{2 + \cos x} dx = \int_0^{2\pi} \frac{2\pi - x + \tan(\sin(2\pi - x))}{2 + \cos(2\pi - x)} dx$$

$$= \int_0^{2\pi} \frac{2\pi - x - \tan(\sin x)}{2 + \cos x} dx \Rightarrow 2\Omega = \int_0^{2\pi} \frac{2\pi}{2 + \cos x} dx \Rightarrow \Omega = \int_0^{2\pi} \frac{\pi}{2 + \cos x} dx$$

$$= \pi \int_0^{2\pi} \frac{1}{1 + (1 + \cos x)} dx = \pi \int_0^{2\pi} \frac{1}{1 + 2\sin^2 \frac{x}{2}} dx = \pi \int_0^{2\pi} \frac{dx}{3\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}}$$

$$= \pi \int_0^{2\pi} \frac{\frac{2dx}{2\cos^2 \frac{x}{2}}}{3\tan^2 \frac{x}{2} + 1} = \pi \times \frac{2}{\sqrt{3}} \left[\tan^{-1}\left(\frac{\tan \frac{x}{2}}{\sqrt{3}}\right)\right]_0^{2\pi} = \frac{2}{\sqrt{3}} \pi^2$$

740. Evaluate:

$$\int_0^1 \frac{2(1 + x^2) + (1 - x)(1 + 3x - x^2 + x^3)\ln(1 + x^2)}{(1 + x^2)^2} e^x dx$$

Proposed by Kunihiko Chikaya-Tokyo-Japan

Solution by Ruangkhaw Chaoka-Chiangrai-Thailand

$$K = \int_0^1 \frac{2(1 + x^2) + (1 - x)(1 + 3x - x^2 + x^3)\ln(1 + x^2)}{(1 + x^2)^2} e^x dx$$

$$I = \int \frac{2(1 + x^2) + (1 - x)(1 + 3x - x^2 + x^3)\ln(1 + x^2)}{(1 + x^2)^2} e^x dx = ??$$

$$\begin{cases} (f(x)e^x)' = \left(f(x) + f'(x)\right)e^x \quad (1) \\ (g(x)h(x)e^x)' = \left[\left(g(x) + g'(x)\right)h(x) + g(x)h'(x)\right]e^x \quad (2) \end{cases}; h(x) = \ln(1 + x^2) \Rightarrow$$

$$\Rightarrow h'(x) = \frac{2x}{1 + x^2}$$

$$g(x) + g'(x) = \frac{(1 - x)(1 + 3x - x^2 + x^3)}{(1 + x^2)^2} = \frac{(1 - x)[(1 + x^2)(x - 1) + 2(x + 1)]}{(1 + x^2)^2} =$$

$$= \frac{-(x-1)^2}{1+x^2} - \frac{2(x^2-1)}{(1+x^2)^2} = \frac{-(x-1)^2}{1+x^2} - \left(\frac{(x-1)^2}{1+x^2}\right)' \Rightarrow g(x) = \frac{-(x-1)^2}{1+x^2}$$

$$\textit{(2)};\ (g(x)h(x)e^x)' = \frac{(1-x)(1+3x-x^2+x^3)\ln(1+x^2)-2x(x-1)^2}{(1+x^2)^2}e^x$$

$$I = \int \frac{(1-x)(1+3x-x^2+x^3)\ln(1+x^2)-2x(x-1)^2}{(1+x^2)^2}e^x dx +$$

$$+\int \frac{2+2x-2x^2+2x^3}{(1+x^2)^2}e^x dx$$

$$\left(f(x)+f'(x)\right) = \frac{2+2x-2x^2+2x^3}{(1+x^2)^2} = \frac{2x(1+x^2)-2(x^2-1)}{(1+x^2)^2} = \frac{2x}{1+x^2} - \frac{2(x^2-1)}{(1+x^2)^2}$$

$$= \frac{2x}{1+x^2} + \left(\frac{2x}{1+x^2}\right)' \Rightarrow f(x) = \frac{2x}{1+x^2}$$

$$\textit{(1)};\ (f(x)e^x)' = \frac{2+2x-2x^2+2x^3}{(1+x^2)^2}e^x$$

$$\therefore I = \frac{2x}{1+x^2}e^x - \frac{(x-1)^2}{1+x^2}\ln(1+x^2)e^x + C \Leftrightarrow \therefore K = e$$

741. Prove without softs:

$$\int_0^{\pi}\left(e^{-\frac{x^2}{\pi^2}} \cdot \sqrt{1+\frac{x^3}{\pi^3}}\right)dx < \pi$$

Proposed by Kunihiko Chikaya-Tokyo-Japan

Solution 1 by Adrian Popa-Romania

$$y = \int_0^{\pi} e^{-\frac{x^2}{\pi^2}}\sqrt{1+\frac{x^3}{\pi^3}}dx < \pi;\ \frac{x}{\pi} = t \Rightarrow \begin{cases} x = 0 \Rightarrow t = 0 \\ x = \pi \Rightarrow t = 1 \end{cases};\ dx = \pi dt$$

$$y = \pi \int_0^1 e^{-t^2}\sqrt{1+t^3}dt < \pi \Leftrightarrow \int_0^1 e^{-t^2}\sqrt{1+t^3}\,dt < 1 \Leftrightarrow$$

$$\Leftrightarrow \left.\begin{matrix}\displaystyle\int_0^1 \frac{\sqrt{1+t^3}}{e^{t^2}}dt < 1 \\ e^{t^2} > t^2+1\end{matrix}\right\} \Rightarrow \int_0^1 \frac{\sqrt{1+t^3}}{e^{t^2}}dt < \int_0^1 \frac{\sqrt{1+t^3}}{1+t^2}dt \overset{?}{<} 1$$

$$\Leftrightarrow \frac{\sqrt{1+t^3}}{1+t^2} \le 1 \ (\forall) t \in (0,1)$$

$$\sqrt{1+t^3} < 1+t^2 \Leftrightarrow 1+t^3 < 1+2t^2+t^4 | : t^2$$

$$t < 2+2t^2 \Leftrightarrow \left.\begin{matrix} t^2-t+2>0 \\ \Delta = 1-8<0 \end{matrix}\right\} \Rightarrow t^2-t+2>0 \ \textbf{\textit{(true)}} \Rightarrow y < \pi$$

Solution 2 by Avishek Mitra-West Bengal-India

$$\Omega = \int_0^\pi e^{-\frac{x^2}{\pi^2}} \sqrt{1+\frac{x^3}{\pi^3}} dx$$

$$\Leftrightarrow \int_0^\pi e^{-\frac{x^2}{\pi^2}} \sqrt{1+\frac{x^3}{\pi^3}} dx \overset{Holder}{\le} \sqrt{\int_0^\pi e^{-\frac{2x^2}{\pi^2}} dx \cdot \int_0^\pi \left(1+\frac{x^3}{\pi^3}\right) dx}$$

$$\Rightarrow \Omega \le \sqrt{\left(\pi + \frac{1}{\pi^3}\cdot\frac{\pi^4}{4}\right)\int_0^\pi e^{-\frac{2x^2}{\pi^2}} dx} \Rightarrow \Omega \le \sqrt{\frac{5\pi}{4}\cdot\frac{\pi}{\sqrt{2}}\int_0^\pi e^{-u^2} du}$$

$$\left[\because \frac{2x^2}{\pi^2} = u^2 \Rightarrow \frac{2}{\pi^2}\cdot 2x\,dx = 2u\,du \Rightarrow \frac{2}{\pi^2}\cdot\frac{\pi}{\sqrt{2}} u\,dx = u\,du \Rightarrow dx = \frac{\pi}{\sqrt{2}} du\right]$$

$$\Rightarrow \Omega \le \sqrt{\frac{5\pi}{4}\cdot\frac{\pi}{\sqrt{2}}\cdot\frac{\sqrt{\pi}}{2}\cdot \mathrm{erf}(\pi)} \quad \left[\because \mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\cdot\int_0^z e^{-t^2} dt\right]$$

$$\Rightarrow \Omega \le \pi\sqrt{\frac{5\sqrt{\pi}}{8\sqrt{2}}\cdot \mathrm{erf}(\pi)} \Leftrightarrow \mathrm{erf}(\pi) < 1 \ [\because 0 \le \mathrm{erf}(x) \le 1 \Leftrightarrow 0 \le x \le \infty]$$

$$\Rightarrow \frac{5\sqrt{\pi}}{8\sqrt{2}}\mathrm{erf}(\pi) < \frac{5\sqrt{\pi}}{8\sqrt{2}} \Rightarrow \sqrt{\frac{8\sqrt{2}}{5\sqrt{\pi}\cdot\mathrm{erf}(\pi)}} > \sqrt{\frac{8\sqrt{2}}{5\sqrt{\pi}}} > 1$$

$$\Leftrightarrow \left[let \sqrt{\frac{8\sqrt{2}}{5\sqrt{\pi}\cdot\mathrm{erf}(\pi)}} = p \Leftrightarrow p > 1\right] \Leftrightarrow p\cdot\Omega \le \pi, \because p > 1 \Leftrightarrow \Omega < \pi$$

Solution 3 by Ravi Prakash-New Delhi-India

$$I = \int_0^\pi e^{-\frac{x^2}{\pi^2}} \sqrt{1+\left(\frac{x}{\pi}\right)^3} dx < \pi$$

Put $\frac{x}{\pi} = t$

$$I = \int_0^1 e^{-t^2}\sqrt{1+t^3}\pi dt$$

For $0 < t < 1, e^{t^2} > 1 + t^2 > 1 + t^3 > \sqrt{1+t^3} \Rightarrow e^{-t^2}\sqrt{1+t^3} < 1$ ***for*** $0 < 1 < 1$

$$\Rightarrow I = \pi\int_0^1 e^{-t^2}\sqrt{1+t^3}\,dt < \pi\int_0^1 dt = \pi$$

742. Prove without softs:

$$\left|\int_0^2 \left(\sqrt{x}\cdot\sin(\pi x)\right)dx\right| < \int_0^1 \left(\left|\sqrt{x+1}-\sqrt{x}\right|\cdot\sin(\pi x)\right)dx$$

Proposed by Daniel Sitaru – Romania

Solution by Adrian Popa – Romania

$$\left|\int_0^2 \sqrt{x}\sin(\pi x)\,dx\right| < \int_0^1 \left(\left|\sqrt{x+1}-\sqrt{x}\right|\sin(\pi x)\right)dx$$

$$\left|\int_0^2 \sqrt{x}\sin(\pi x)dx\right| = \left|\int_0^1 \sqrt{x}\sin(\pi x)\,dx + \underbrace{\int_1^2 \sqrt{x}\sin(\pi x)\,dx}_{y}\right| \quad \textbf{(1)}$$

$$\left.\begin{array}{r} x-1 = t \Rightarrow \begin{array}{l} x = t+1 \\ dx = dt\end{array} \\ x = 1 \Rightarrow t = 0 \\ x = 2 \Rightarrow t = 1 \end{array}\right\} \Rightarrow \left.\begin{array}{c} y = \int_0^1 \sqrt{t+1}\sin(\pi(t+1))dt \\ \sin(\pi t + \pi) = \sin\pi t\underbrace{\cos\pi}_{-1} + \underbrace{\sin\pi}_{0}\cos(\pi t) = -\sin\pi t \end{array}\right\} \Rightarrow$$

$$\Rightarrow y = \int_0^1 -\sqrt{t+1}\sin(\pi t)\,dt \text{ or } y = -\int_0^1 \sqrt{x+1}\sin\pi x\,dx$$

$$\textbf{(1)}\Rightarrow \left|\int_0^2 \sqrt{x}\sin(\pi x)\,dx\right| = \left|\int_0^1 \sqrt{x}\sin(\pi x)\,dx - \int_0^1 \sqrt{x+1}\sin(\pi x)\,dx\right| =$$

$$= \left|\int_0^1 \left(\sqrt{x}-\sqrt{x+1}\sin\pi x\,dx\right)\right| \le \int_0^1 \left|\left(\sqrt{x}-\sqrt{x+1}\right)\sin(\pi x)\right|dx =$$

$$= \left.\begin{array}{c}\displaystyle\int_0^1 \left|\sqrt{x+1}-\sqrt{x}\right|\cdot|\sin(\pi x)|dx \\ x\in[0;1]\Rightarrow \pi x\in[0,\pi]\Rightarrow \sin(\pi x)>0\Rightarrow|\sin(\pi x)|=\sin\pi x\end{array}\right\}\Rightarrow$$

$$\Rightarrow\left|\int_0^2 \sqrt{x}\sin(\pi x)\,dx\right| < \int_0^1 \left|\sqrt{x+1}-\sqrt{x}\right|\sin\pi x\,dx$$

743. If $0 < a \le b < \frac{\pi}{2}$ then:

$$\int_a^b \left(\frac{\tan x}{x\cos x}\right)dx \ge \log\left(\frac{1+\tan b}{1+\tan a}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\because \tan x > x \therefore \int_a^b \frac{\tan x\,dx}{x\cos x} > \int_a^b \sec x\,dx$$

$$(\therefore x,\cos x>0) = \log\left(\frac{\sec b+\tan b}{\sec a+\tan a}\right) \overset{?}{\ge} \log\left(\frac{1+\tan b}{1+\tan a}\right)$$

$$\Leftrightarrow \frac{\sec b+\tan b}{\sec a+\tan a} \overset{?}{\ge} \frac{1+\tan b}{1+\tan a}\ (\because f(\theta)=\log\theta\ is\uparrow\forall\theta\in\mathbb{R})$$

$$\Leftrightarrow \sec b+\tan b+\tan a\sec b \overset{?}{\ge} \sec a+\tan a+\tan b\sec a$$

$$\Leftrightarrow \tan b(\sec b-\sec a)+\tan a(\sec b-\sec a) \overset{?}{\ge} 0$$

$$\Leftrightarrow \sec b-\sec a \overset{?}{\ge} 0\ \ (\because \tan a+\tan b>0)$$

$$\Leftrightarrow \frac{1}{\cos b} \overset{?}{\ge} \frac{1}{\cos a} \Leftrightarrow \cos a \overset{?}{\ge} \cos b \to \textbf{\textit{true}} \because 0 < a \le b < \frac{\pi}{2}$$

$$\therefore \int_a^b \frac{\tan x\,dx}{x\cos x} \ge \log\left(\frac{1+\tan b}{1+\tan a}\right)$$

Solution 2 by Mokhtar Khassani-Mostaganem-Algerie

$$\therefore f(x)=\sin x+\sin x\tan x-x;\ f'(x)=\cos x+\sin x+\frac{\sin x}{\cos^2 x}-1=$$

$$=\sqrt{2}\cos\left(\frac{\pi}{4}-x\right)+\frac{\sin x}{\cos^2 x}-1>0, f(0)=0, f(x)\nearrow \forall x\in\left]0,\frac{\pi}{2}\right[\Rightarrow\frac{\sin x}{x}\geq\frac{1}{1+\tan x}$$

$$\therefore\int_a^b\frac{\tan x}{x\cos x}dx=\int_a^b\frac{\sin x}{x\cos^2 x}dx\geq\int_a^b\frac{d(\tan x)}{1+\tan x}=\log\left(\frac{1+\tan b}{1+\tan a}\right)$$

744. Prove without softs:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\left(\frac{1}{(\sin x)^{\sin x}}+\frac{1}{(\cos x)^{\cos x}}\right)dx<\frac{\pi}{3}-\frac{\sqrt{3}-1}{2}$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaijan

Solution by Tran Hong-Dong Thap-Vietnam

$$\Leftrightarrow\frac{1}{(\sin x)^{\sin x}}=(\csc x)^{\sin x}=\big(1+(\csc x-1)\big)^{\sin x}\overset{Bernoulli}{\leq}1+\sin x\,(\csc x-1)$$

$$=1+1-\sin x=2-\sin x$$

$$\Leftrightarrow\frac{1}{(\csc x)^{\cos x}}=(\sec x)^{\cos x}=\big(1+(\sec x-1)\big)^{\cos x}\overset{Bernoulli}{\leq}1+\cos(\sec x-1)$$

$$=1+1-\cos x=2-\cos x$$

$$\Leftrightarrow\frac{1}{(\sin x)^{\sin x}}+\frac{1}{(\cos x)^{\cos x}}\leq 4-(\sin x+\cos x)$$

$$\Rightarrow\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\left(\frac{1}{(\sin x)^{\sin x}}+\frac{1}{(\cos x)^{\cos x}}\right)dx\leq\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\big(4-(\sin x+\cos x)\big)\,dx$$

$$=4\left(\frac{\pi}{4}-\frac{\pi}{6}\right)+\left(\cos\frac{\pi}{4}-\cos\frac{\pi}{6}\right)-\left(\sin\frac{\pi}{4}-\sin\frac{\pi}{6}\right)$$

$$=4\cdot\frac{3\pi-2\pi}{12}+\left(\frac{1}{\sqrt{2}}-\frac{\sqrt{3}}{2}\right)-\left(\frac{1}{\sqrt{2}}-\frac{1}{2}\right)=\frac{\pi}{3}+\frac{1}{2}-\frac{\sqrt{3}}{2}=\frac{\pi}{3}-\frac{\sqrt{3}-1}{2}$$

745. If $0 < a \le b < 2\pi$ then:

$$\frac{1}{6(b-a)^2}\int_a^b\int_a^b\int_a^b\left(\frac{\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{y+z}{2}\right)\sin\left(\frac{z+x}{2}\right)}{\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}\sin\left(\frac{x+y+z}{2}\right)}+\cot\left(\frac{x+y+z}{2}\right)\right)dx\,dy\,dz \le \log\left|\frac{b}{2\sin\frac{a}{2}}\right|$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Mokhtar Kassani-Mostaganem-Algerie

$$\int_a^b\int_a^b\int_a^b\left(\frac{\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x+y}{2}\right)}{\sin\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right)\sin\left(\frac{z}{2}\right)\sin\left(\frac{x+y+z}{2}\right)}+\cot\left(\frac{x+y+z}{2}\right)\right)dx\,dy\,dz =$$

$$= \int_a^b\int_a^b\int_a^b\left(\cot\left(\frac{x}{2}\right)+\cot\left(\frac{y}{2}\right)+\cot\left(\frac{z}{2}\right)\right)dx\,dy\,dz$$

$$= \int_a^b\int_a^b\left(2\log\left|\frac{\sin\left(\frac{b}{2}\right)}{\sin\left(\frac{a}{2}\right)}\right|+(b-a)\cot\left(\frac{y}{2}\right)+(b-a)\cot\left(\frac{z}{2}\right)\right)dy\,dz$$

$$= \int_a^b\left(2(b-a)\log\left|\frac{\sin\left(\frac{b}{2}\right)}{\sin\left(\frac{a}{2}\right)}\right|+2(b-a)\log\left|\frac{\sin\left(\frac{b}{2}\right)}{\sin\left(\frac{a}{2}\right)}\right|+(b-a)^2\cot\left(\frac{z}{2}\right)\right)dy$$

$$= 6(b-a)^2\log\left|\frac{\sin\left(\frac{b}{2}\right)}{\sin\left(\frac{a}{2}\right)}\right| \le 6(b-a)^2\log\left|\frac{b}{2\sin\left(\frac{a}{2}\right)}\right|$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

With $0 < x, y, z < 2\pi$

Let $\Omega(x,y,z) = \frac{\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{y+z}{2}\right)\sin\left(\frac{x+z}{2}\right)}{\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}\sin\frac{x+y+z}{2}}+\cot\frac{x+y+z}{2} = \frac{\sin\left(\frac{x+y}{2}\right)+\sin\left(\frac{y+z}{2}\right)+\sin\left(\frac{x+z}{2}\right)}{\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}\sin\frac{x+y+z}{2}}+$

$$+\frac{\cos\frac{x+y+z}{2}}{\sin\frac{x+y+z}{2}} = \frac{\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{y+z}{2}\right)\sin\left(\frac{x+z}{2}\right)+\cos\left(\frac{x+y+z}{2}\right)\cdot\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}}{\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}\sin\frac{x+y+z}{2}} =$$

$$= \frac{\left(\sin\frac{x+y+z}{2}\right)\left(\sin\frac{y}{2}\sin\frac{z}{2}\cos\frac{x}{2}+\sin\frac{x}{2}\sin\frac{z}{2}\cos\frac{y}{2}+\sin\frac{x}{2}\sin\frac{y}{2}\cos\frac{z}{2}\right)}{\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}\sin\frac{x+y+z}{2}}$$

$$=\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}+\frac{\cos\frac{y}{2}}{\sin\frac{y}{2}}+\frac{\cos\frac{z}{2}}{\sin\frac{z}{2}}\to\frac{1}{6(b-a)^2}\int_a^b\int_a^b\int_a^b\Omega(x,y,z)\,dx\,dy\,dz$$

$$=\frac{1}{6(b-a)^2}\int_a^b\int_a^b\int_a^b\left(\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}+\frac{\cos\frac{y}{2}}{\sin\frac{y}{2}}+\frac{\cos\frac{z}{2}}{\sin\frac{z}{2}}\right)dx\,dy\,dz=\omega$$

$$\int_a^b\int_a^b\int_a^b\left(\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\right)dx\,dy\,dz=\left(\int_a^b\int_a^b dx\,dy\right)\left(\int_a^b\frac{\cos\frac{z}{2}}{\sin\frac{z}{2}}dz\right)=$$

$$=2(b-a)^2\left(\ln\sin\frac{b}{2}-\ln\sin\frac{a}{2}\right)=2(b-a)^2\left(\ln\frac{\sin\frac{b}{2}}{\sin\frac{a}{2}}\right)$$

$$\int_a^b\int_a^b\int_a^b\left(\frac{\cos\frac{y}{2}}{\sin\frac{y}{2}}\right)dx\,dy\,dz=\left(\int_a^b\int_a^b dx\,dy\right)\left(\int_a^b\frac{\cos\frac{y}{2}}{\sin\frac{y}{2}}dy\right)=$$

$$=2(b-a)^2\left(\ln\sin\frac{b}{2}-\ln\sin\frac{a}{2}\right)=2(b-a)^2\left(\ln\frac{\sin\frac{b}{2}}{\sin\frac{a}{2}}\right)$$

$$\int_a^b\int_a^b\int_a^b\left(\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\right)dx\,dy\,dz=\left(\int_a^b\int_a^b dy\,dz\right)\left(\int_a^b\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}dx\right)=$$

$$=2(b-a)^2\left(\ln\sin\frac{b}{2}-\ln\sin\frac{a}{2}\right)=2(b-a)^2\left(\ln\frac{\sin\frac{b}{2}}{\sin\frac{a}{2}}\right)$$

$$\to\omega=\ln\frac{\sin\frac{b}{2}}{\sin\frac{a}{2}}\overset{\sin\frac{b}{2}\le\frac{b}{2};b>0}{\le}\ln\frac{\frac{b}{2}}{\sin\frac{a}{2}}=\ln\frac{b}{2\sin\frac{a}{2}}$$

746. Prove without softs:

$$\int_0^{\frac{1}{4}}\int_0^{\frac{1}{4}}\int_0^{\frac{1}{4}}\left(\frac{\sin(\pi x)+\sin(\pi y)+\sin(\pi z)}{x+y+z}\right)dxdydz>\frac{\sqrt{2}}{32}$$

Proposed by Jalil Hajimir-Canada

Solution by Daniel Sitaru-Romania

$$x,y,z\in\left[0,\frac{1}{4}\right]\to\pi x,\pi y,\pi z\in\left[0,\frac{\pi}{4}\right]\to sin(\pi x)\geq\frac{2\sqrt{2}}{\pi}\cdot\pi x=2\sqrt{2}x$$

(J.SANDOR'S INEQUALITY-2005-VICTORIA UNIVERSITY)

Analogous: $\sin(\pi y)\geq 2\sqrt{2}y,\sin(\pi z)\geq 2\sqrt{2}z$

$$\sin(\pi x)+\sin(\pi y)+\sin(\pi z)\geq 2\sqrt{2}(x+y+z)$$

Equality holds for $x=y=z=0$. **If** $x+y+z\neq 0\to$

$$\frac{\sin(\pi x)+\sin(\pi y)+\sin(\pi z)}{x+y+z}>2\sqrt{2}$$

$$\int_0^{\frac{1}{4}}\int_0^{\frac{1}{4}}\int_0^{\frac{1}{4}}\left(\frac{sin(\pi x)+sin(\pi y)+sin(\pi z)}{x+y+z}\right)dxdydz>2\sqrt{2}\int_0^{\frac{1}{4}}\int_0^{\frac{1}{4}}\int_0^{\frac{1}{4}}dxdydz=2\sqrt{2}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}=\frac{\sqrt{2}}{32}$$

747. Prove that:

$$\int_0^1\int_0^1\int_0^1\left(\frac{xyz}{5}+e^{-\frac{xyz}{4}}\right)dxdydz<1$$

Proposed by Jalil Hajimir-Canada

Solution by Daniel Sitaru-Romania

$$f:[0,1]X[0,1]X[0,1]\to\mathbb{R},f(x,y,z)=\frac{xyz}{5}+e^{-\frac{xyz}{4}}$$

$$f'_x=\frac{yz}{5}-\frac{1}{4}e^{-\frac{xyz}{4}},f''_{xx}=\frac{1}{16}e^{-\frac{xyz}{4}},f''_{yy}=\frac{1}{16}e^{-\frac{xyz}{4}},f''_{zz}=\frac{1}{16}e^{-\frac{xyz}{4}}$$

f convexe in each variable on a compact set (a convexe cube with sides parallel with axis). By Weierstrass-Gireaux theorem f has an attained maximum in a vertex of cube $maxf(x,y,z)=max\begin{Bmatrix}f(0,0,0),f(1,0,0),f(0,1,0),f(0,0,1),\\f(1,1,0),f(1,0,1),f(0,1,1),f(1,1,1)\end{Bmatrix}=max\left\{1,\frac{1}{5}+\frac{1}{\sqrt[4]{e}}\right\}=1$

$$f(x,y,z)=\frac{xyz}{5}+e^{-\frac{xyz}{4}}\leq 1\to\int_0^1\int_0^1\int_0^1 f(x,y,z)dxdydz<\int_0^1\int_0^1\int_0^1 dxdydz=1\to$$

$$\int_0^1\int_0^1\int_0^1\left(\frac{xyz}{5}+e^{-\frac{xyz}{4}}\right)dxdydz<1$$

748. If $a,b,c>0$ then:

$$\frac{1}{abc}\int_a^{2a}\int_b^{2b}\int_c^{2c}\left(\frac{x^3+y^3+z^3}{(2x+y+z)(x+2y+z)(x+y+2z)}\right)dxdydz>\frac{3}{64}$$

Proposed by Dilshodbek Akramov-Azerbaijan

Solution by Daniel Sitaru-Romania

$$x^3+y^3+z^3=\frac{x^3}{1^2}+\frac{y^3}{1^2}+\frac{z^3}{1^2}\overset{RADON}{\geq}\frac{(x+y+z)^3}{(1+1+1)^2}=\frac{(x+y+z)^3}{9}$$

$$\frac{1}{\prod_{cyc}(2x+y+z)}\overset{AM-GM}{\geq}\frac{1}{\left(\frac{4x+4y+4z}{3}\right)^3}=\frac{27}{64(x+y+z)^3}$$

$$\frac{x^3+y^3+z^3}{\prod_{cyc}(2x+y+z)}\geq\frac{(x+y+z)^3}{9}\cdot\frac{27}{64(x+y+z)^3}=\frac{3}{64}$$

$$\int_a^{2a}\int_b^{2b}\int_c^{2c}\frac{x^3+y^3+z^3}{\prod_{cyc}(2x+y+z)}dxdydz>\int_a^{2a}\int_b^{2b}\int_c^{2c}\frac{3}{64}dxdydz=\frac{3}{64}abc$$

$$\frac{1}{abc}\int_a^{2a}\int_b^{2b}\int_c^{2c}\frac{x^3+y^3+z^3}{\prod_{cyc}(2x+y+z)}dxdydz>\frac{3}{64}$$

749. If $f:[a,b]\to(0,1),\ a,b\in\mathbb{R},\ a\leq b,\ f$ – continuous then:

$$\int_a^b\int_a^b\int_a^b\left(\prod_{cyc}(\sin f(x)\cdot\sin^{-1}f(x))\right)dx\,dy\,dz\geq\left(\int_a^b f^2(x)\,dx\right)^3$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Jalil Hajimir-Canada

By using the fact that $\sin t \le t \le \sin^{-1} t$ ***for*** $0 \le t \le 1$ ***and*** $f(x) = \frac{\sin x}{x}$ ***is increasing on*** $[0,1]$***. We can conclude:*** $\frac{\sin t}{t} \ge \frac{\sin(\sin^{-1} t)}{\sin^{-1} t}$; $0 \le t \le 1$

$$0 \le t \le 1 \quad (*)$$

Which means: $\sin t \cdot \sin^{-1} t \ge t^2$***. If we call the triple integral*** I***:***

$$I = \left(\int_a^b \sin f(x) \cdot \sin^{-1} f(x)\, dx\right)^3 \overset{(*)}{\ge} \left(\int_a^b f^2(x)\, dx\right)^3$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

Because: $f(x)$: ***continuous on*** $[a;b]$, $t = f(x) \in (0,1)$***, we let:***

$$g(t) = (\sin t \cdot \sin^{-1} t) - t^2, 0 < t < 1 \to g'(t) = \frac{\sin t}{\sqrt{1-t^2}} - 2t + \sin t \cdot \sin^{-1} t =$$

$$= \frac{\sin t - 2t\sqrt{1-t^2} + \sin t \cdot \sin^{-1} t \cdot \sqrt{1-t^2}}{\sqrt{1-t^2}} = 0 \leftrightarrow t = 0 \notin (0,1)$$

t	0		1
$g'(t)$	0	$+$	$\Vert$
$g(t)$	0	$\nearrow$	$+\infty$

Hence: $g(t) > 0 \leftrightarrow \sin t \cdot \sin^{-1} t > t^2, 0 < t < 1$

$$\to \int_a^b\int_a^b\int_a^b \left(\prod_{cyc} \sin f(x) \cdot \sin^{-1} f(x)\right) dx\,dy\,dz \ge \int_a^b\int_a^b\int_a^b \left(f^2(x) \cdot f^2(y) \cdot f^2(z)\right) dx\,dy\,dz$$

$$= \left(\int_a^b f^2(x)\, dx\right)\left(\int_a^b f^2(y)\, dy\right)\left(\int_a^b f^2(z)\right) = \left(\int_a^b f^2(x)\, dx\right)^3$$

Proved. Equality if and only if $a = b$***.***

750. Prove that:

$$\int_0^{\frac{\pi}{2}} (1+\sin x)^{\sin x}\,dx < \frac{3\pi}{4}$$

Proposed by Jalil Hajimir-Canada

Solution 1 by Daniel Sitaru-Romania

$$f:[0,\infty)\to\mathbb{R},\ f(t)=1+tx-(1+t)^x,\ f(0)=0,\ x\in(0,1)-fixed$$

$$f'(t)=x\left(1-\frac{1}{(1+t)^{1-x}}\right)$$

$$x<1\to 1-x>0,\ (1+t)^{1-x}>(1+t)^0=1\to 1>\frac{1}{(1+t)^{1-x}}\to$$

$$1-\frac{1}{(1+t)^{1-x}}>0\to f'(t)>0\to f-increasing,\ f(t)>f(0)=0$$

$$(1+t)^x<1+tx,\ t>0,\ x\in(0,1);\ (1+\sin x)^{\sin x}<1+\sin x\cdot\sin x$$

$$\int_0^{\frac{\pi}{2}}(1+\sin x)^{\sin x}dx<\int_0^{\frac{\pi}{2}}(1+\sin x\cdot\sin x)dx=\frac{\pi}{2}+\int_0^{\frac{\pi}{2}}\left(\frac{1-\cos 2x}{2}\right)dx=$$

$$=\frac{\pi}{2}+\frac{\pi}{4}-\frac{1}{4}(\sin\pi-\sin 0)=\frac{3\pi}{4}$$

Solution 2 by Mokhtar Khassani-Mostaganem-Algerie

$$\int_0^{\frac{\pi}{2}}(1+\sin x)^{\sin x}dx\overset{BERNOULLI}{\lesssim}\int_0^{\frac{\pi}{2}}(1+\sin x\cdot\sin x)dx=\frac{\pi}{2}+\int_0^{\frac{\pi}{2}}\left(\frac{1-\cos 2x}{2}\right)dx=$$

$$=\frac{\pi}{2}+\frac{\pi}{4}-\frac{1}{4}(\sin\pi-\sin 0)=\frac{3\pi}{4}$$

751. $\rho(x)=\left|\left[x+\frac{1}{2}\right]-x\right|$, $[*]$ - great integer function. Prove that:

$$\left(\int_0^1\left(\sqrt[3]{\rho(x)}\right)dx\right)\left(\int_0^1\left(\sqrt[5]{\rho(x)}\right)dx\right)\left(\int_0^1\left(\sqrt[7]{\rho(x)}\right)dx\right)<1$$

Proposed by Daniel Sitaru – Romania

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Solution 1 by Adrian Popa-Romania

$$\rho(x)=\left|\left[x+\frac{1}{2}\right]-x\right|\Rightarrow \begin{array}{l} I.\,If\ x\in\left[0;\frac{1}{2}\right)\Rightarrow \rho(x)=|0-x|=x \\ II.\,If\ x\in\left[\frac{1}{2};1\right]\Rightarrow \rho(x)=|1-x|=1-x \end{array}$$

$$\int_0^1 \sqrt[3]{\rho(x)}\,dx=\int_0^{\frac{1}{2}}\sqrt[3]{x}\,dx+\underbrace{\int_{\frac{1}{2}}^1 \sqrt[3]{1-x}\,dx}_{\left.\begin{array}{c}1-x=t\Rightarrow -dx=dt\\ x'=\frac{1}{2}\Rightarrow t=\frac{1}{2}\\ x=1\Rightarrow t=0\end{array}\right\}\Rightarrow}$$

$$\Rightarrow\int_0^1\sqrt[3]{\rho(x)}\,dx=\int_0^{\frac{1}{2}}\sqrt[3]{x}dx+\int_0^{\frac{1}{2}}\sqrt[3]{t}\,dt=2\int_0^{\frac{1}{2}}\sqrt[3]{x}\,dx=2\cdot\left.\frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right|_0^{\frac{1}{2}}=$$

$$=2\cdot\frac{3}{4}\cdot\sqrt[3]{\left(\frac{1}{2}\right)^4}=2\cdot\frac{1}{2}\cdot\frac{3}{4}\sqrt[3]{\frac{1}{2}}=\frac{3}{4}\sqrt[3]{\frac{1}{2}}$$

$$\int_0^1\sqrt[5]{\rho(x)}\,dx=\int_0^{\frac{1}{2}}\sqrt[5]{x}\,dx+\int_{\frac{1}{2}}^1\sqrt[5]{1-x}\,dx=2\int_0^{\frac{1}{2}}\sqrt[5]{x}\,dx=2\cdot\left.\frac{x^{\frac{6}{5}}}{\frac{6}{5}}\right|_0^{\frac{1}{2}}=$$

$$=2\cdot\frac{5}{6}\sqrt[5]{\left(\frac{1}{2}\right)^6}=2\cdot\frac{1}{2}\cdot\frac{5}{6}\cdot\sqrt[5]{\frac{1}{2}}=\frac{5}{6}\sqrt[5]{\frac{1}{2}}$$

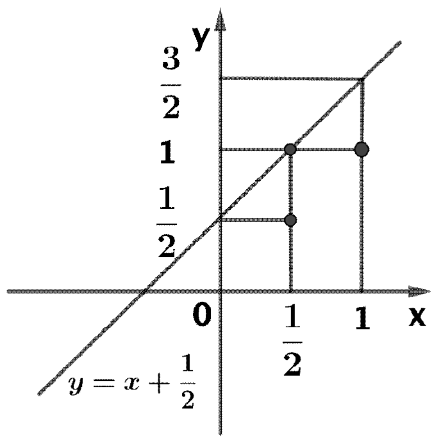
$$\int_0^1\sqrt[7]{\rho(x)}\,dx=\int_0^{\frac{1}{2}}\sqrt[7]{x}\,dx+\int_{\frac{1}{2}}^1\sqrt[7]{1-x}\,dx=2\int_0^{\frac{1}{2}}\sqrt[7]{x}\,dx=2\cdot\left.\frac{x^{\frac{8}{7}}}{\frac{8}{7}}\right|_0^{\frac{1}{2}}=$$

$$=2\cdot\frac{7}{8}\cdot\sqrt[7]{\left(\frac{1}{2}\right)^8}=2\cdot\frac{7}{8}\cdot\frac{1}{2}\sqrt[7]{\frac{1}{2}}=\frac{7}{8}\sqrt[7]{\frac{1}{2}}$$

$$\left(\int_0^1\sqrt[3]{\rho(x)}\,dx\right)\left(\int_0^1\sqrt[5]{\rho(x)}\,dx\right)\left(\int_0^1\sqrt[7]{\rho(x)}\,dx\right)=\frac{3}{4}\sqrt[3]{\frac{1}{2}}\cdot\frac{5}{6}\sqrt[5]{\frac{1}{2}}\cdot\frac{7}{8}\sqrt[7]{\frac{1}{2}}=$$

$$= \frac{35}{64}\sqrt[3]{\frac{1}{2}}\sqrt[5]{\frac{1}{2}}\sqrt[7]{\frac{1}{2}} = \frac{35}{2^6}\sqrt[3]{\frac{1}{2}}\sqrt[5]{\frac{1}{2}}\sqrt[7]{\frac{1}{2}} = \frac{35}{2^6\cdot 2^{\frac{1}{3}}\cdot 2^{\frac{1}{5}}\cdot 2^{\frac{1}{7}}} < \frac{35}{2^6} = \frac{35}{64} < 1$$

Solution 2 by Khaled Abd Imouti-Damascus-Syria



x	$-\infty$	$\frac{1}{2}$	1	$+\infty$
$1-x$	$+ + + + + + + + + + + 0 - - - - - -$			

$$\rho(x) = \left|\left[x+\frac{1}{2}\right] - x\right|$$

$$0 \le x \le 1;\ \frac{1}{2} \le x+\frac{1}{2} \le \frac{3}{2};\ \left[x+\frac{1}{2}\right] - x = \begin{cases} |-x|: 0 \le x < \frac{1}{2} \\ |1-x|: \frac{1}{2} \le x < 1 \\ 0: x = 1 \end{cases}$$

$$\rho(x) = \begin{cases} x: 0 \le x < \frac{1}{2} \\ 1-x: \frac{1}{2} \le x < 1 \\ 0: x = 1 \end{cases};\ \int_0^1 \sqrt[3]{\rho(x)}\,dx = \int_0^{\frac{1}{2}} \sqrt[3]{x}\,dx + \int_{\frac{1}{2}}^1 \sqrt[3]{1-x}\,dx$$

$$\int_0^1 \sqrt[3]{\rho(x)}\,dx = \frac{3}{2}\int_0^1 \sqrt[3]{\frac{1}{16}} = \frac{3}{2}\int_0^1 \sqrt[3]{\frac{1}{8}\cdot\frac{1}{2}};\ \int_0^1 \sqrt[3]{\rho(x)}\,dx = \frac{3}{2}\cdot\frac{1}{8}\sqrt[3]{\frac{1}{2}} = \frac{3}{4}\cdot\sqrt[3]{\frac{1}{2}}$$

$$\int_0^1 \sqrt[5]{\rho(x)}\,dx = \int_0^{\frac{1}{2}} \sqrt[5]{x}\,dx + \int_{\frac{1}{2}}^1 \sqrt[5]{1-x}\,dx$$

$$\int_0^1 \sqrt[5]{\rho(x)}\,dx = \frac{5}{6}\sqrt[5]{\frac{1}{64}} + \frac{5}{6}\sqrt[5]{\frac{1}{64}} = \frac{10}{6}\sqrt[5]{\frac{1}{32\cdot 2}} = \frac{5}{3}\cdot\frac{1}{2}\sqrt[5]{\frac{1}{2}}$$

$$\int_0^1 \sqrt[5]{\rho(x)}\,dx = \frac{5}{6}\sqrt[5]{\frac{1}{2}};\ \int_0^1 \sqrt[7]{\rho(x)}\,dx = \frac{7}{8}\sqrt[7]{\frac{1}{2}}$$

So: $\left(\int_0^1 \sqrt[3]{\rho(x)}\,dx\right)\left(\int_0^1 \sqrt[5]{\rho(x)}\,dx\right)\left(\int_0^1 \sqrt[7]{\rho(x)}\,dx\right) = \frac{1}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\sqrt[3]{\frac{1}{2}}\cdot\sqrt[5]{\frac{1}{2}}\cdot\sqrt[7]{\frac{1}{2}}$

$$= \frac{35}{64}\left(\frac{1}{2}\right)^{\frac{1}{3}+\frac{1}{5}+\frac{1}{7}} = \frac{35}{64}\left(\frac{1}{2}\right)^{\frac{71}{105}} < \frac{35}{64} < 1$$

$$\int_0^{\frac{1}{2}} x^{\frac{1}{3}}\,dx = \left[\frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right]_0^{\frac{1}{2}} = \frac{3}{4}\left[\sqrt[3]{x^4}\right]_0^{\frac{1}{2}} = \frac{3}{4}\left[\sqrt[3]{\frac{1}{16}}\right]$$

$$-\int_{\frac{1}{2}}^1 -(1-x)^{\frac{1}{3}}\,dx = -\left[\frac{(1-x)^{\frac{4}{3}}}{\frac{4}{3}}\right]_{\frac{1}{2}}^1 = -\frac{3}{4}\left[\sqrt[3]{(1-x)^4}\right]_{\frac{1}{2}}^1 = -\frac{3}{4}\left[(0) - \sqrt[3]{\frac{1}{16}}\right] = +\frac{3}{4}\sqrt[3]{\frac{1}{16}}$$

$$\int_0^{\frac{1}{2}} x^{\frac{1}{5}}\,dx = \left[\frac{x^{\frac{6}{5}}}{\frac{6}{5}}\right]_0^{\frac{1}{2}} = \frac{5}{6}\left[\sqrt[5]{x^6}\right]_0^{\frac{1}{2}} = \frac{5}{6}\left[\sqrt[5]{\frac{1}{64}}\right]$$

$$-\int_{\frac{1}{2}}^1 -(1-x)^{\frac{1}{5}}\,dx = -\left[\frac{(1-x)^{\frac{6}{5}}}{\frac{6}{5}}\right]_{\frac{1}{2}}^1 = -\frac{5}{6}\left[\sqrt[5]{(1-x)^6}\right]_{\frac{1}{2}}^1 = -\frac{5}{6}\left[0 - \sqrt[5]{\frac{1}{64}}\right] = \frac{5}{6}\sqrt[5]{\frac{1}{64}}$$

Note: $\int_0^1 \sqrt[5]{\rho(x)}\,dx = \left(\frac{n}{n+1}\right)\sqrt[n]{\frac{1}{2}}$

Solution 3 by Jalil Hajimir-Canada

Using the following facts:

1. $0 \le x \le 1$ ***then*** $\int_0^1 \sqrt[n]{x} < 1$

2. $0 \le \rho(x) \le 1$

We have: $\int_0^1 \sqrt[n]{\rho(x)}\,dx < 1 \therefore \left(\int_0^1 \sqrt[3]{\rho(x)}\,dx\right)\left(\int_0^1 \sqrt[5]{\rho(x)}\,dx\right)\left(\int_0^1 \sqrt[7]{\rho(x)}\,dx\right) < 1$

752. Prove without softs:

$$\int_0^1 \sqrt{\frac{\sin\left(\frac{\pi}{2}\right)x}{x+1}}\,dx < \ln 2$$

Proposed by Jalil Hajimir-Canada

Solution 1 by Avishek Mitra-West Bengal-India

$$\Leftrightarrow \int_0^1 \sqrt{\frac{\sin\left(\frac{\pi x}{2}\right)}{(x+1)}}\,dx \overset{Holder}{\le} \sqrt{\int_0^1 \sin\left(\frac{\pi x}{2}\right)dx \cdot \int_0^1 \frac{dx}{(x+1)}}$$

$$\Rightarrow \Omega \le \sqrt{\frac{2}{\pi}\left(1-\cos\frac{\pi}{2}\right)\cdot[\ln(x+1)]_0^1} = \sqrt{\frac{2}{\pi}\ln 2}$$

$\Leftrightarrow$ *(*)* $\sqrt{\frac{2}{\pi}\ln 2} < \ln 2 \Rightarrow \frac{2}{\pi}\ln 2 < \ln^2 2 \Rightarrow \frac{2}{\pi} < \ln 2$ ***((*) true as*** $\pi \approx 3.14, \ln 2 \approx 0.693$***)***

$$\Leftrightarrow \int_0^1 \sqrt{\frac{\sin\left(\frac{\pi x}{2}\right)}{(x+1)}}\,dx < \ln 2$$

Solution 2 by Adrian Popa-Romania

$$\left(\int_a^b f(x)\cdot g(x)dx\right)^2 \le \left(\int_a^b f^2(x)\,dx\right)\left(\int_a^b g^2(x)\,dx\right) \overset{C.B.S.}{\Rightarrow}$$

$$\Rightarrow \int_a^b f(x)\cdot g(x)dx \le \sqrt{\int_a^b f^2(x)\,dx \cdot \int_a^b g^2(x)\,dx}$$

We take $f(x) = \sqrt{\sin\frac{\pi x}{2}}$; $g(x) = \sqrt{\frac{1}{x+1}}$; $a = 0; b = 1$

$$\int_0^1 \sqrt{\frac{\sin\frac{\pi x}{2}}{x+1}}\,dx \le \sqrt{\int_0^1 \sin\frac{\pi x}{2}\,dx \cdot \int_0^1 \frac{1}{x+1}\,dx} =$$

$$= \sqrt{-\frac{2}{\pi}\cos\frac{\pi x}{2}\Big|_0^1 \cdot \ln(x+2)|_1^2} = \sqrt{-\left(0-\frac{2}{\pi}\right)\cdot \ln 2} = \sqrt{\frac{2}{\pi}\ln 2}$$

We must prove that: $\sqrt{\frac{2}{\pi}\ln 2} < \ln 2 \Leftrightarrow \frac{2}{\pi}\ln 2 < \ln^2 2 \,|: \ln 2$

$$\Leftrightarrow \frac{2}{\pi} < \ln 2 \Leftrightarrow 2 < \ln 2^{\pi} \Leftrightarrow \ln e^2 < \ln 2^{\pi} \Leftrightarrow e^2 < 2^{\pi}$$

$e^2 < 2.8^2 = 7.84 < 8 < 2^3 < 2^{\pi} \Rightarrow e^2 < 2^{\pi}$ ***. Q.E.D.***

753. Prove without softs:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \log(\tan x)\,dx > \frac{\pi\log\sqrt{3}}{3} + \log\sqrt{\frac{7-4\sqrt{3}}{3-2\sqrt{2}}}$$

Proposed by Seyran Ibrahimov-Sumgait-Azerbaijan

Solution by Avishek Mitra-West Bengal-India

$$\Rightarrow \Omega = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \log(\tan x)\,dx = [x\cdot\log(\tan x)]_{\frac{\pi}{4}}^{\frac{\pi}{3}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{x\cdot\sec^2 x}{\tan x}\,dx$$

$$= \frac{\pi}{3}\log\sqrt{3} - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{x}{\sin x\cdot\cos x}\,dx = \frac{\pi}{3}\log\sqrt{3} + \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{x}{\sin x\cdot\cos x}\,dx$$

$$\Leftrightarrow \frac{x}{\cos x} \ge 1 \left[\because x \in \left(\frac{\pi}{4},\frac{\pi}{3}\right)\right] \Rightarrow \frac{x}{\sin x\cdot\cos x} \ge \csc x$$

$$\Rightarrow \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{x}{\sin x\cdot\cos x}\,dx \ge \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \csc x\,dx = \left[\log\left(\tan\frac{x}{2}\right)\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \log\left(\tan\frac{\pi}{8}\right) - \log\left(\tan\frac{\pi}{6}\right)$$

$$\Rightarrow \frac{\pi}{3}\log\sqrt{3} + I \geq \frac{\pi}{3}\log\sqrt{3} + \log\left(\tan\frac{\pi}{12}\right) - \log\left(\tan\frac{\pi}{8}\right) -$$

$$-\left[\log\left(\tan\frac{\pi}{12}\right) + \log\left(\tan\frac{\pi}{6}\right) - 2\log\left(\tan\frac{\pi}{8}\right)\right] \Leftrightarrow \Omega \geq \frac{\pi}{3}\log\sqrt{3} + \log\left(\frac{\tan\frac{\pi}{12}}{\tan\frac{\pi}{8}}\right) + p$$

$$\Leftrightarrow \left[\begin{array}{c} \because \tan\frac{\pi}{12} = \tan 15 = \tan(45-30) = \frac{1-\tan 30}{1+\tan 30} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3}, \\ \tan\frac{\pi}{4} = \frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}} \Rightarrow \frac{2x}{1-x^2} = 1 \Rightarrow x^2 + 2x - 1 = 0 \Rightarrow \\ \Rightarrow \frac{-2 \pm \sqrt{4 - 4\cdot 1\cdot(-1)}}{2} = -1 \pm \sqrt{2} \Rightarrow \tan\frac{\pi}{8} = \sqrt{2} - 1 \Rightarrow \\ \Rightarrow \log\left(\frac{\tan\frac{\pi}{12}\cdot\tan\frac{\pi}{6}}{\tan^2\frac{\pi}{8}}\right) = \log\left(\frac{(2-\sqrt{3})\frac{1}{\sqrt{3}}}{\left(\sqrt{2}-1\right)^2}\right) = \log\left(\frac{2-\sqrt{3}}{(3-2\sqrt{2})\sqrt{3}}\right) < 0 \\ \because 0 < \frac{2-\sqrt{3}}{(3-2\sqrt{2})\sqrt{3}} < 1 \Rightarrow \\ \Rightarrow \log\left(\tan\frac{\pi}{12}\right) + \log\left(\tan\frac{\pi}{6}\right) - 2\log\left(\tan\frac{\pi}{8}\right) = -p \end{array}\right]$$

$$\Leftrightarrow \Omega \geq \frac{\pi}{3}\log\sqrt{3} + \log\left(\frac{2-\sqrt{3}}{\sqrt{2}-1}\right) + p = \frac{\pi}{3}\log\sqrt{3} + \log\sqrt{\frac{\left(2-\sqrt{3}\right)^2}{\left(\sqrt{2}-1\right)^2}} + p =$$

$$= \frac{\pi}{3}\log\sqrt{3} + \log\sqrt{\frac{7-4\sqrt{3}}{3-2\sqrt{2}}} + p \Leftrightarrow \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \log(\tan x)\,dx > \frac{\pi}{3}\log\sqrt{3} + \log\sqrt{\frac{7-4\sqrt{3}}{3-2\sqrt{2}}}$$

754. If $0 < a, b, c < 2\pi$ then:

$$\frac{1}{(a+b+c)^3}\int_{\frac{a}{2}}^{a}\int_{\frac{b}{2}}^{b}\int_{\frac{c}{2}}^{c}\left(\frac{\sin\frac{x}{2}\sin\frac{y}{2} + \sin\frac{z}{2}\sin\left(\frac{x+y+z}{2}\right)}{\sin\left(\frac{x+z}{2}\right)\sin\left(\frac{y+z}{2}\right)}\right)dxdydz \leq \frac{1}{216}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sin\frac{x}{2}\sin\frac{y}{2}+\sin\frac{z}{2}\sin\left(\frac{x+y+z}{2}\right)=$$

$$=\frac{1}{2}\left(\cos\frac{x-y}{2}-\cos\frac{x+y}{2}+\cos\frac{x+y}{2}-\cos\frac{x+y+2z}{2}\right)$$

$$=\frac{1}{2}\cdot 2\sin\left(\frac{\frac{x-y}{2}+\frac{x+y+2z}{2}}{2}\right)\sin\left(\frac{\frac{x+y+2z}{2}-\frac{x-y}{2}}{2}\right)=\sin\left(\frac{x+z}{2}\right)\sin\left(\frac{y+z}{2}\right)$$

$$\therefore LHS=\frac{1}{(a+b+c)^3}\int_{\frac{a}{2}}^{a}\int_{\frac{b}{2}}^{b}\int_{\frac{c}{2}}^{c}\frac{\sin\left(\frac{x+z}{2}\right)\sin\left(\frac{y+z}{2}\right)}{\sin\left(\frac{x+z}{2}\right)\sin\left(\frac{y+z}{2}\right)}dxdydz=\frac{abc}{8(a+b+c)^3}\overset{A-G}{\le}\frac{abc}{8\cdot 27abc}=\frac{1}{216}\ \textit{(Proved)}$$

Solution 2 by Remus Florin Stanca-Romania

$$\sin\frac{x}{2}\sin\frac{y}{2}=\left(\cos\left(\frac{x-y}{2}\right)-\cos\left(\frac{x+y}{2}\right)\right)\frac{1}{2},$$

$$\sin\frac{z}{2}\sin\left(\frac{x+y+z}{2}\right)=\frac{1}{2}\left(\cos\frac{x+y}{2}-\cos\left(\frac{x+y+2z}{2}\right)\right)\ \textit{and}$$

$$\sin\left(\frac{x+z}{2}\right)\sin\left(\frac{y+z}{2}\right)=\frac{1}{2}\left(\cos\frac{x-y}{2}-\cos\frac{x+y+2z}{2}\right)\Rightarrow$$

$$\Rightarrow\frac{\sin\frac{x}{2}\sin\frac{y}{2}+\sin\frac{z}{2}\sin\left(\frac{x+y+z}{2}\right)}{\sin\left(\frac{x+z}{2}\right)\sin\left(\frac{y+z}{2}\right)}=$$

$$=\frac{\cos\left(\frac{x-y}{2}\right)-\cos\left(\frac{x+y}{2}\right)+\cos\left(\frac{x+y}{2}\right)-\cos\left(\frac{x+y+2z}{2}\right)}{\cos\left(\frac{x-y}{2}\right)-\cos\left(\frac{x+y+2z}{2}\right)}=1\Rightarrow$$

$$\Rightarrow\int_{\frac{a}{2}}^{a}\int_{\frac{b}{2}}^{b}\int_{\frac{c}{2}}^{c}\left(\frac{\sin\frac{x}{2}\sin\frac{y}{2}+\sin\frac{z}{2}\sin\left(\frac{x+y+z}{2}\right)}{\sin\left(\frac{x+z}{2}\right)\sin\left(\frac{y+z}{2}\right)}\right)dxdydz=\frac{abc}{8}$$

We need to prove that $\frac{abc}{8(a+b+c)^3}\le\frac{1}{216}$

$$a+b+c\ge 3\sqrt[3]{abc}\Rightarrow(a+b+c)^3\ge 27abc\Rightarrow\frac{abc}{8(a+b+c)^3}\le\frac{1}{27\cdot 8}=\frac{1}{216}\ \textit{(Proved)}$$

$$\Rightarrow\frac{1}{(a+b+c)^3}\int_{\frac{a}{2}}^{a}\int_{\frac{b}{2}}^{b}\int_{\frac{c}{2}}^{c}\left(\frac{\sin\frac{x}{2}\sin\frac{y}{2}+\sin\frac{z}{2}\sin\left(\frac{x+y+z}{2}\right)}{\sin\left(\frac{x+z}{2}\right)\sin\left(\frac{y+z}{2}\right)}\right)dxdydz\le\frac{1}{216}\ \textit{(Q.E.D.)}$$

755. Prove:

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sqrt{\frac{\cos^8 xz}{\cos^2 yz}+\frac{\sin^8 xz}{\sin^2 yz}+\frac{\cos^8 yz}{\sin^2 xz}+\frac{\sin^8 yz}{\cos^2 xz}}\,dzdydx \geq \frac{\sqrt{2}}{16}\pi^3$$

Proposed by Jalil Hajimir-Canada

Solution 1 by Daniel Sitaru-Romania

$$a = xz \in \left[0,\frac{\pi^2}{4}\right] \subset \left[0,\frac{3\pi}{2}\right], \sin^2 2a \leq 1 \to 1-4\sin^2 a\cos^2 a \geq 0 \to 1-2\sin^2 a\cos^2 a \geq \frac{1}{2} \to \sin^4 a+\cos^4 a \geq \frac{1}{2}$$

$$b = yz \in \left[0,\frac{\pi^2}{4}\right] \subset \left[0,\frac{3\pi}{2}\right], \sin^2 2b \leq 1 \to 1-4\sin^2 b\cos^2 b \geq 0 \to 1-2\sin^2 b\cos^2 b \geq \frac{1}{2} \to$$

$$\to \sin^4 b+\cos^4 b \geq \frac{1}{2} \to \sin^4 a+\cos^4 a+\sin^4 b+\cos^4 b \geq 1$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sqrt{\frac{\cos^8 a}{\cos^2 b}+\frac{\sin^8 a}{\sin^2 b}+\frac{\cos^8 b}{\sin^2 a}+\frac{\sin^8 b}{\cos^2 a}}\,dxdydz \overset{BERGSTROM}{\geq}$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sqrt{\frac{(\sin^4 a+\cos^4 a+\sin^4 b+\cos^4 b)^2}{2}}\,dxdydz \geq \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sqrt{\frac{1}{2}}\,dxdydz =$$

$$= \frac{\sqrt{2}}{2}\cdot\left(\frac{\pi}{2}\right)^3 = \frac{\sqrt{2}}{16}\pi^3$$

Solution 2 by Ali Jaffal-Lebanon

$$I = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sqrt{\frac{\cos^8 xz}{\cos^2 yz}+\frac{\sin^8 xz}{\sin^2 yz}+\frac{\cos^8 yz}{\sin^2 xz}+\frac{\sin^8 yz}{\cos^2 xz}}\,dzdydx \geq \frac{\sqrt{2}}{16}\pi^3$$

Let $a = \cos xz$; $b = \cos yz$; $c = \sin yz$ ***and*** $d = \sin xz$

$$\frac{\cos^8 xz}{\cos^2 yz}+\frac{\sin^8 xz}{\sin^2 yz}+\frac{\cos^8 yz}{\sin^2 xz}+\frac{\sin^8 yz}{\cos^2 xz} = \frac{a^8}{b^2}+\frac{d^8}{c^2}+\frac{b^8}{d^2}+\frac{c^8}{a^2}$$

and $a^2+d^2 = 1, b^2+c^2 = 1$

We have $\frac{a^8}{b^2}+\frac{d^8}{c^2}+\frac{b^8}{d^2}+\frac{c^8}{a^2} \geq \frac{(a^4+d^4+b^4+c^4)^2}{b^2+c^2+d^2+a^2} \geq \frac{(a^4+b^4+c^4+d^4)^2}{2}$

but $a^4+b^4+c^4+d^4 = \frac{1}{4}(a^2+b^2+c^2+d^2)^2 = \frac{1}{4}(4) = 1$

So, $\frac{a^8}{b^2}+\frac{d^8}{c^2}+\frac{b^8}{d^2}+\frac{c^8}{a^2}\ge\frac{1}{2}$; $I\ge\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sqrt{\frac{1}{2}}dxdydz\ge\frac{\pi^3}{8}\times\frac{\sqrt{2}}{2}$

Therefore $I\ge\frac{\pi^3\sqrt{2}}{16}$

756. If $0<a,b<1$ then:

$$\Omega=\int_{-a}^{a}\int_{-b}^{b}\left(\frac{e^{x^2+y^2-2}}{(a^x+1)(b^x+1)}\right)dxdy<1$$

Proposed by Jalil Hajimir-Canada

GENERALIZATION

If $0<a_1,a_2,\dots,a_n<1, n\in\mathbb{N}, n\ge 1$ then:

$$\Omega=\int_{-a_1}^{a_1}\int_{-a_2}^{a_2}\dots\int_{-a_n}^{a_n}\left(\frac{e^{x_1^2+x_2^2+\dots+x_n^2-n}}{(a_1^x+1)\cdot(a_2^x+1)\cdot\dots\cdot(a_n^x+1)}\right)dx_1dx_2\dots dx_n<1$$

Daniel Sitaru

Solution by Daniel Sitaru-Romania

$$I_1=\int_{-a}^{a}\frac{e^{x^2}}{a^x+1}dx=\int_{a}^{-a}\frac{e^{(-x)^2}}{a^{-x}+1}(-dx)=\int_{-a}^{a}\frac{e^{x^2}\cdot a^x}{a^x+1}dx$$

$$2I_1=\int_{-a}^{a}\frac{e^{x^2}\cdot(a^x+1)}{a^x+1}dx=\int_{-a}^{a}e^{x^2}dx<e\big(a-(-a)\big)=2ae$$

$$I_1<ae;\ I_2=\int_{-b}^{b}\frac{e^{x^2}}{b^x+1}dx<be$$

$$\Omega=\int_{-a}^{a}\int_{-b}^{b}\left(\frac{e^{x^2+y^2-2}}{(a^x+1)(b^x+1)}\right)dxdy=\frac{1}{e^2}\left(\int_{-a}^{a}\frac{e^{x^2}}{a^x+1}dx\right)\left(\int_{-b}^{b}\frac{e^{y^2}}{b^y+1}dy\right)=$$

$$=\frac{1}{e^2}\cdot I_1\cdot I_2<\frac{1}{e^2}\cdot ae\cdot be=ab<1$$

757. If $a,b,c>0, a+b+c=3$ then:

$$\int_0^a \left(\frac{e^x}{2}\right)^{e^x} dx+\int_0^b \left(\frac{e^x}{2}\right)^{e^x} dx+\int_0^c \left(\frac{e^x}{2}\right)^{e^x} dx>3e-6$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Adrian Popa-Romania

$$\int_0^a \left(\frac{e^x}{2}\right)^{e^x} dx=\int_0^a \left(1+\frac{e^x}{2}-1\right)^{e^x} dx\overset{Bernoulli}{\geq}\int_0^a 1+e^x\left(\frac{e^x}{2}-1\right)dx=$$

$$=\int_0^a \left(1-e^x+\frac{e^{2x}}{2}\right)dx=x\Big|_0^a-e^x\Big|_0^a+\frac{e^{2x}}{4}\Big|_0^a=$$

$$=\left.\begin{matrix} a-e^a+1+\dfrac{e^{2a}}{4}-\dfrac{1}{4}=a-\dfrac{1}{4}+\left(\dfrac{e^a}{2}-1\right)^2 \\ a>0\Rightarrow e^a>1\Rightarrow\left(\dfrac{e^a}{2}-1\right)^2>\left(\dfrac{1}{2}-1\right)^2=\dfrac{1}{4}\end{matrix}\right\}\Rightarrow\int_0^a \left(\frac{e^x}{2}\right)^{e^x} dx>a-\frac{1}{4}+\frac{1}{4}=a$$

Similar: $\int_0^b \left(\frac{e^x}{2}\right)^{e^x} dx>b$ **and** $\int_0^c \left(\frac{e^x}{2}\right)^{e^x} dx>c$

$$\int_0^a \left(\frac{e^x}{2}\right)^{e^x} dx+\int_0^b \left(\frac{e^x}{2}\right)^{e^x} dx+\int_0^c \left(\frac{e^x}{2}\right)^{e^x} dx>a+b+c=3\overset{?}{>}3e-6\Leftrightarrow 9>3e\ \textit{(true)}$$

Note: $\begin{cases}\dfrac{e^x}{2}-1>-1\ because\ x\in(0;a)\ and\ a>0\\ e^x>1\ because\ x>0\end{cases}$

Solution 2 by Abdul Mukhtar-Nigeria

$$\Omega=\int_0^a \left(\frac{e^x}{2}\right)^{e^x} dx+\int_0^b \left(\frac{e^x}{2}\right)^{e^x} dx+\int_0^c \left(\frac{e^x}{2}\right)^{e^x} dx>3e-6$$

$\Omega=I_1+I_2+I_3$. **For** $I_1=\int_0^a \left(\frac{e^x}{2}\right)^{e^x} dx;\ I_1=\int_0^a \left(1+\frac{e^x}{2}-1\right)^{e^x} dx$

Using Bernoulli $(1+x)^n>1+nx$

$$I_1=\int_0^a 1+e^x\left(\frac{e^x}{2}-1\right)dx;\ I_1=\int_0^a 1+\frac{e^{2x}}{2}-e^x dx$$

$$I_1 = \int_0^a dx + \frac{1}{2}\int_0^a e^{2x}\,dx - \int_0^a e^x\,dx;\ I_1 = \left[x + \frac{1}{4}e^{2x} - e^x\right]_0^a$$

$$I_2 = \left[a + \frac{1}{4}e^{2a} - \frac{1}{4}e^0 - e^a + e^0\right];\ I_1 = \left[a + \frac{1}{4}e^{2a} - \frac{1}{4} - e^a + 1\right]$$

But $a > 0$; $I_1 = (a + 0 + 0) = a \therefore I_1 = \int_0^a \left(\frac{e^x}{2}\right)^{e^x} dx = a$

For I_2 we use the same method from I_1 we get:

$$I_2 = \left[x + \frac{1}{4}e^x - e^x\right]_0^b;\ I_2 = \left[b + \frac{1}{4}e^{2b} - \frac{1}{4}e^0 - e^b + e^0\right]$$

$I_2 = \left[b + \frac{1}{4}e^{2b} - \frac{1}{4} - e^b + 1\right]$. **But** $b > 0$; $I_2 = (b + 0 + 0) = b \therefore I_2 = \int_0^b \left(\frac{e^x}{2}\right)^{e^x} dx = b$

Now, to I_3 we use the same idea from I_1 and I_2 to get:

$$I_3 = \int_0^c \left(\frac{e^x}{2}\right)^{e^x} dx = c \therefore \Omega = a + b + c$$

But, $a + b + c = 3$; $\Omega = a + b + c = 3 > 3e - 6$; $\Omega = 3 > 3e - 6$; $\Omega = 3 > 2.15485$

$$\therefore \Omega = \int_0^a \left(\frac{e^x}{2}\right)^{e^x} dx + \int_0^b \left(\frac{e^x}{2}\right)^{e^x} dx + \int_0^c \left(\frac{e^x}{2}\right)^{e^x} dx = a + b + c = 3 > 3e - 6$$

Solution 3 by Ali Jaffal-Lebanon

$$I = \int_0^a \left(\frac{e^x}{2}\right)^{e^x} dx + \int_0^b \left(\frac{e^x}{2}\right)^{e^x} dx + \int_0^c \left(\frac{e^x}{2}\right)^{e^x} dx$$

Let $I_a = \int_0^a \left(\frac{e^x}{2}\right)^{e^x} dx$; $I_b = \int_0^b \left(\frac{e^x}{2}\right)^{e^x} dx$ ***and*** $I_c = \int_0^c \left(\frac{e^x}{2}\right)^{e^x} dx$

We have $\left(\frac{e^x}{2}\right)^{e^x} = e^{(x-\ln 2)e^x} \geq (x - \ln 2)e^x + 1$

Since $e^t \geq t + 1$ ***for all*** $t \in \mathbb{R}$

Then $I_a \geq (a - \ln 2 - 1)e^a + \ln 2 + 1 + a$

So, $I \geq (a - \ln 2 - 1)e^a + (a - \ln 2 - 1)e^b + (c - \ln 2 - 1)e^c + 3\ln 2 + 3 + a + b + c$

We have $\varphi(x) = (x - \ln 2 - 1)e^x$ ***is convex or*** $[0, +\infty]$

So, $\varphi\left(\frac{a+b+c}{3}\right) \leq \frac{\varphi(a)+\varphi(b)+\varphi(c)}{3}$

Then $I \geq 3\left(\frac{a+b+c}{3} - \ln 2 - 1\right)e^{\frac{a+b+c}{3}} + 3\ln 2 + 3 + 3$

$$I \geq 3[-(\ln 2)e + \ln 2 + 2] \geq 3e - 6$$

758. Prove without softs:

$$\left(\int_0^{\frac{1}{2}} 2^{-x^2} dx\right)^{-1} + \left(\int_{\frac{1}{2}}^{1} 2^{-x^2} dx\right)^{-1} > 4$$

Proposed by Jalil Hajimir-Canada

Solution 1 by Daniel Sitaru-Romania

$$x^2 > 0, -x^2 < 0, 2^{-x^2} < 1, \int_0^1 2^{-x^2} dx < 1$$

$$\frac{1}{\int_0^1 2^{-x^2} dx} > 1 \quad (1)$$

$$\left(\int_0^{\frac{1}{2}} 2^{-x^2} dx\right)^{-1} + \left(\int_{\frac{1}{2}}^{1} 2^{-x^2} dx\right)^{-1} =$$

$$= \frac{1}{\int_0^{\frac{1}{2}} 2^{-x^2} dx} + \frac{1}{\int_{\frac{1}{2}}^{1} 2^{-x^2} dx} \overset{BERGSTROM}{\geq} \frac{(1+1)^2}{\int_0^{\frac{1}{2}} 2^{-x^2} dx + \int_{\frac{1}{2}}^{1} 2^{-x^2} dx} = \frac{4}{\int_0^1 2^{-x^2} dx} \overset{(1)}{>} \frac{4}{1} = 4$$

Solution 2 by Adrian Popa – Romania

$$\underbrace{\left(\int_0^{\frac{1}{2}} 2^{-x^2}\, dx\right)}_{a}^{-1} + \underbrace{\left(\int_{\frac{1}{2}}^{1} 2^{-x^2}\, dx\right)}_{b}^{-1} \geq 4 \Rightarrow a^{-1} + b^{-1} \geq 4$$

$$\frac{2}{\frac{1}{a}+\frac{1}{b}} \overset{MH<MA}{\leq} \frac{a+b}{2} \Rightarrow \left(\frac{1}{a}+\frac{1}{b}\right)(a+b) \geq 4 \Rightarrow \frac{1}{a}+\frac{1}{b} \geq \frac{4}{a+b}$$

$$a+b = \int_0^{\frac{1}{2}} 2^{-x^2}\,dx + \int_{\frac{1}{2}}^{1} 2^{-x^2}\,dx = \int_0^1 2^{-x^2}\,dx \overset{?}{\leq} 1 \Leftrightarrow 2^{-x^2} \leq 1 \ (\forall)x \in [0,1]$$

$$f(x) = 2^{-x^2} \Rightarrow f'(x) = -2x \cdot 2^{-x^2}\ln 2 = 0 \Rightarrow x = 0$$

x	0 $\qquad\qquad\qquad\qquad\qquad$ 1
$f'(x)$	0 – – – – – – – – – – – – – – – – –
$f(x)$	1 $\qquad\qquad\qquad\qquad\qquad$ $\frac{1}{2}$

$$\Rightarrow f(x) \leq 1 \ (\forall)x \in (0,1) \Rightarrow 2^{-x^2} \leq 1$$

$$\left(\int_0^{\frac{1}{2}} 2^{-x^2}\,dx\right)^{-1} + \left(\int_{\frac{1}{2}}^{1} 2^{-x^2}\,dx\right)^{-1} \geq \frac{4}{\int_0^1 2^{-x^2}\,dx} \geq 4$$

759. Prove that:

$$\frac{\pi+4}{\pi-4} + \int_1^a \frac{(\tan^{-1}x)^2}{(x-\tan^{-1}x)^2}\,dx > \frac{1+\sin a \cdot \tan^{-1}a}{\tan^{-1}a - a}, a > 1$$

Proposed by Daniel Sitaru – Romania

Solution by Tran Hong-Dong Thap-Vietnam

First, we compute: $I(a) = \int_1^a \frac{(\tan^{-1}x)^2}{(x-\tan^{-1}x)^2}\,dx$

We have: $f(x) = \frac{(\tan^{-1}x)^2}{(x-\tan^{-1}x)^2} = \frac{(\tan^{-1}x)^2}{x^2-2x\tan^{-1}x+(\tan^{-1}x)^2} = 1 - \frac{x(x-2\tan^{-1}x)}{(x-\tan^{-1}x)^2}$

More, $g(x) = \frac{x^2+1}{\tan^{-1}x-x} \to g'(x) = \frac{2x(\tan^{-1}x-x)+(x^2+1)\cdot\frac{x^2}{x^2+1}}{(x-\tan^{-1}x)^2} = -\frac{x(x-2\tan^{-1}x)}{(x-\tan^{-1}x)^2}$

$\to \int\left(-\frac{x(x-2\tan^{-1}x)}{(x-\tan^{-1}x)^2}\right)dx = g(x) + C = \frac{x^2+1}{\tan^{-1}x-x} + C \ (C: const)$***. Hence:***

$$I(a)=\int_1^a \frac{(\tan^{-1}x)^2}{(x-\tan^{-1}x)^2}dx=\int_1^a f(x)\,dx=\int_1^a\left(1-\frac{x(x-2\tan^{-1}x)}{(x-\tan^{-1}x)^2}\right)dx$$

$$=\left(x+\frac{x^2+1}{\tan^{-1}x-x}\right)\Big|_1^a=\left(a+\frac{a^2+1}{\tan^{-1}a-a}\right)-\left(1+\frac{2}{\tan^{-1}1-1}\right)$$

$$=\frac{a\tan^{-1}a+1}{\tan^{-1}a-a}-\frac{\tan^{-1}1+1}{\tan^{-1}1-1}=\frac{a\tan^{-1}a+1}{\tan^{-1}a-a}-\frac{\pi+4}{\pi-4}$$

$$\to LHS=\frac{\pi+4}{\pi-4}+I(a)=\frac{\pi+4}{\pi-4}+\frac{a\tan^{-1}a+1}{\tan^{-1}a-a}-\frac{\pi+4}{\pi-4}=\frac{a\tan^{-1}a+1}{\tan^{-1}a-a}$$

$$=\frac{1+a\tan^{-1}a}{\tan^{-1}a-a}\overset{a>\sin a, a>1}{>}\frac{1+\sin a\tan^{-1}a}{\tan^{-1}a-a}.$$ ***Proved.***

760. a **and** b **are two real numbers such that** $1<a<b<e^2$**.**

Prove that:

$$\int_a^b\int_a^b\int_a^b \frac{x+y+z}{\ln\left(\frac{x+y+z}{3}\right)}dx\,dy\,dz\ge\frac{3(b-a)^3(a+b)}{2\ln\left(\frac{a+b}{2}\right)}$$

Proposed by Ali Jaffal – Lebanon

Solution 1 by Daniel Sitaru – Romania

$$f:(1,\infty)\to\mathbb{R}, f(x)=\frac{x}{\ln x}, f'(x)=\frac{\ln x-1}{(\ln x)^2},$$

$$f''(x)=\frac{2-\ln x}{(\ln x)^3}>0, \forall x\in(1,e^2)$$

f –convexe. By Hermite-Hadamard's inequality for triple integrals:

$$\frac{1}{(b-a)^3}\int_a^b\int_a^b\int_a^b f\left(\frac{x+y+z}{3}\right)dxdydz\ge f\left(\frac{a+b}{2}\right)$$

$$\int_a^b\int_a^b\int_a^b\left(\frac{\frac{x+y+z}{3}}{\ln\left(\frac{x+y+z}{3}\right)}\right)dxdy\ge\frac{(b-a)^3(a+b)}{2\ln\left(\frac{a+b}{2}\right)}$$

$$\int_a^b\int_a^b\int_a^b\left(\frac{x+y+z}{\ln\left(\frac{x+y+z}{3}\right)}\right)dxdy\geq\frac{3(b-a)^3(a+b)}{2\ln\left(\frac{a+b}{2}\right)}$$

Solution 2 by proposer

Let f be a convex and continuous function on $[a;b]$ then:

$$\int_a^b f(x)\,dx\geq(b-a)f\left(\frac{a+b}{2}\right)$$

Let $a\leq x\leq b;a\leq y\leq b$ and $a\leq z\leq b$, so $\int_a^b f\left(\frac{x+y+z}{3}\right)dx\geq(b-a)f\left(\frac{a+b}{6}+\frac{y}{3}+\frac{z}{3}\right)$. And

$$\int_a^b\int_a^b f\left(\frac{x+y+z}{3}\right)dx\,dy\geq\int_a^b(b-a)\,f\left(\frac{a+b}{6}+\frac{y}{3}+\frac{z}{3}\right)dy\geq(b-a)^2f\left(\frac{a+b}{6}+\frac{a+b}{6}+\frac{z}{3}\right)$$

Then: $\int_a^b\int_a^b\int_a^b f\left(\frac{x+y+z}{3}\right)dx\,dy\,dz\geq(b-a)^3f\left(\frac{a+b}{6}+\frac{a+b}{6}+\frac{a+b}{6}\right)\geq(b-a)^3f\left(\frac{a+b}{2}\right)$

If $f(x)=\frac{x}{\ln(x)}$ *where* $x\in]1;e^2[$ *then* $f''(x)=\frac{-\ln(x)+2}{x(\ln(x))^3}>0$ *on* $]1;e^2[$

We have f is convex and continuous on $]1;e^2[$

$$\int_a^b\int_a^b\int_a^b\frac{x+y+z}{\ln\left(\frac{x+y+z}{3}\right)}dx\,dy\,dz\geq3(b-a)^3f\left(\frac{a+b}{2}\right)=\frac{3(b-a)^3(a+b)}{2\ln\left(\frac{a+b}{2}\right)}$$

Therefore $\int_a^b\int_a^b\int_a^b\frac{x+y+z}{\ln\left(\frac{x+y+z}{3}\right)}dx\,dy\,dz\geq\frac{3(b-a)^3(a+b)}{2\ln\left(\frac{a+b}{2}\right)}$

761. If $a,b\in\mathbb{R},a\leq b$ then:

$$\int_a^b\int_a^b\sqrt[16]{e^{(3x+y)^2}}\,dxdy\geq(b-a)^2\cdot\sqrt[4]{e^{(a+b)^2}}$$

Proposed by Daniel Sitaru-Romania

Solution by Ali Jaffal-Lebanon

$$\varphi_y(x)=e^{\frac{(3x+y)^2}{16}},\varphi_y''(x)=\frac{9}{8}e^{\frac{(3x+y)^2}{16}}+\frac{9}{64}(3x+y)^2e^{\frac{(3x+y)^2}{16}}\geq0$$

$$\varphi_y - convexe \to \int_a^b \varphi_y(x)dx \ge (b-a)\varphi_y\left(\frac{a+b}{2}\right) = (b-a)e^{\frac{\left(3\left(\frac{a+b}{2}\right)+y\right)^2}{16}}$$

$$\textit{Let } \Psi(y) = exp\left(\frac{3(a+b)}{8} + \frac{y}{4}\right)^2$$

$$\Psi''(y) = \frac{1}{8}exp\left(\frac{3(a+b)}{8} + \frac{y}{4}\right)^2 + \frac{1}{4}\left(\frac{3(a+b)}{8} + \frac{y}{4}\right)^2 exp\left(\frac{3(a+b)}{8} + \frac{y}{4}\right)^2$$

$$\Psi''(y) > 0, \Psi - convexe \to \int_a^b \Psi(y)dy \ge (b-a)\Psi\left(\frac{a+b}{2}\right)$$

$$\int_a^b\int_a^b \sqrt[16]{e^{(3x+y)^2}}dxdy \ge (b-a)^2 exp\left(\frac{3(a+b)}{8} + \frac{a+b}{8}\right)^2 =$$

$$= (b-a)^2 exp\left(\frac{(a+b)^2}{4}\right) = (b-a)^2 \cdot \sqrt[4]{e^{(a+b)^2}}$$

762. If $\frac{\sqrt{3}}{3} < a \le b < 1$ then:

$$\int_a^b\int_a^b \left(\frac{x+y}{\tan^{-1}\left(\frac{x+y}{2}\right)}\right)dx\,dy \ge \frac{(b^2-a^2)(b-a)}{2\tan^{-1}\left(\frac{a+b}{2}\right)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ali Jaffal-Lebanon

Let $\varphi(t) = \frac{t}{\arctan(t)}$ ***where*** $t > 0$; $\varphi''(t) = \frac{2(t-\arctan t)}{(\arctan t)^3(1+t^2)^2} > 0$

Since $t \ge \arctan t$ ***for all*** $t \in [0,+\infty[$ ***we have*** φ ***is convexe on*** $]0,+\infty[$

then $\int_a^b \varphi(t)dt \ge (v-u)\varphi\left(\frac{v+u}{2}\right)$ ***for all*** $0 < u < v$

Let $I = \int_a^b\int_a^b \frac{x+y}{\arctan\left(\frac{x+y}{2}\right)}dx\,dy = 2\int_a^b\int_a^b \varphi\left(\frac{x+y}{2}\right)dx\,dy$

$$\int_a^b \varphi\left(\frac{x+y}{2}\right)dx \ge (b-a)\varphi\left(\frac{b+a}{4} + \frac{y}{2}\right)$$

then

$$I \geq 2\int_a^b (b-a)\varphi\left(\frac{b+a}{4}+\frac{y}{2}\right)dy \geq 2(b-a)(b-a)\varphi\left(\frac{b+a}{4}+\frac{b+a}{4}\right)$$

$$\geq 2(b-a)(b-a)\cdot\varphi\left(\frac{b+a}{2}\right) \geq 2(b-a)(b-a)\times\frac{\frac{b+a}{2}}{\tan^{-1}\left(\frac{b+a}{2}\right)}$$

$$\geq \frac{(b-a)(b^2-a^2)}{\tan^{-1}\left(\frac{a+b}{2}\right)} \geq \frac{(b-a)(b^2-a^2)}{2\tan^{-1}\left(\frac{a+b}{2}\right)}$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

With $\frac{\sqrt{3}}{3} < a \leq b < 1$ ***and*** $x, y \in [a; b] \to a \leq \frac{x+y}{2} \leq b$***. We have:***

$$f(t) = \frac{t}{\tan^{-1} t}, t > 0 \to f'(t) = \frac{1}{\tan^{-1} t} - \frac{t}{(t^2+1)(\tan^{-1} t)^2} = \frac{\tan^{-1} t - \frac{t}{(t^2+1)}}{(\tan^{-1} t)^2}$$

Let: $g(t) = \tan^{-1} t - \frac{t}{(t^2+1)}, t > 0 \to g'(t) = \frac{2t^2}{(t^2+1)^2} > 0 \to g(t) \uparrow$ ***on*** $(0; +\infty) \to$

$$g(t) > g(0) = 0 \to f'(t) > 0 \to f(t) \uparrow \textbf{ on } (0; +\infty)$$

Choosing: $t = \frac{x+y}{x} \geq a > 0 \to \frac{\frac{x+y}{x}}{\tan^{-1}\left(\frac{x+y}{2}\right)} \geq \frac{a}{\tan^{-1}(a)}$

We must show that: $\frac{a}{\tan^{-1}(a)} \geq \frac{b+a}{4\tan^{-1}\left(\frac{a+b}{2}\right)} \leftrightarrow 4a\tan^{-1}\left(\frac{a+b}{2}\right) \geq (b+a)\tan^{-1}(a)$

It is true because: $\tan^{-1} x \uparrow$ ***on*** $(0, +\infty) \overset{\frac{a+b}{2}>0}{\to} \tan^{-1}\left(\frac{a+b}{2}\right) \geq \tan^{-1}(a)$

And: $4a \geq b + a \leftrightarrow 3a \overset{(*)}{\geq} b$

$a > \frac{\sqrt{3}}{3} \to 3a > \sqrt{3} > 1 > b$***. So,***

$$\int_a^b\int_a^b \left(\frac{\frac{x+y}{2}}{\tan^{-1}\left(\frac{x+y}{2}\right)}\right)dx\,dy \geq \left(\frac{b+a}{4\tan^{-1}\left(\frac{a+b}{2}\right)}\right)\int_a^b\int_a^b dx\,dy = \frac{(b-a)^2(b+a)}{4\tan^{-1}\left(\frac{a+b}{2}\right)}$$

Or

$$\int_a^b\int_a^b \left(\frac{x+y}{\tan^{-1}\left(\frac{x+y}{2}\right)}\right)dx\,dy \geq \frac{(b^2-a^2)(b-a)}{2\tan^{-1}\left(\frac{a+b}{2}\right)}$$

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763. If $a, b > 1$ then:

$$\frac{2}{(a-1)(b-1)}\int_1^a\int_1^a\left(\frac{\tan^{-1}x-\tan^{-1}y}{\log x-\log y}\right)dx\,dy<1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Jalil Hajimir-Canada

Let $f(x)=\tan^{-1}x$ *and* $g(x)=\tan x$

Functions $f(x)$ *and* $g(x)$ *are satisfying generalized mean value theorem:*

$$\frac{f(x)-f(y)}{g(x)-g(y)}=\frac{f'(c)}{g'(c)}=\frac{c}{1+c^2}<\frac{1}{2}$$

$$\therefore\int_1^a\int_1^b\frac{f(x)-f(y)}{g(x)-g(y)}dx\,dy<\frac{1}{2}(b-1)(a-1)$$

Solution 2 by Ravi Prakash-New Delhi-India

For $x\neq y, 1<x<a, 1<y<b$*. By Cauchy's mean value theorem:*

$\frac{\tan^{-1}x-\tan^{-1}y}{\log x-\log y}=\frac{\frac{1}{(1+c^2)}}{\frac{1}{c}}$*. For the same* c *lying between* x *and* y*.*

$$\Rightarrow\frac{\tan^{-1}x-\tan^{-1}y}{\log x-\log y}\leq\frac{c}{1+c^2}<\frac{1}{2}\quad[\because c>1]\Rightarrow\int_1^a\int_1^b\frac{\tan^{-1}x-\tan^{-1}y}{\log x-\log y}dx\,dy$$

$$<\frac{1}{2}\int_1^a\int_1^b dx\,dy=\frac{1}{2}(a-1)(b-1)\Rightarrow\frac{2}{(a-1)(b-1)}\int_1^a\int_1^b\frac{\tan^{-1}x-\tan^{-1}y}{\log x-\log y}dx\,dy<1$$

Solution 3 by Florentin Vișescu-Romania

Considering the functions $f,g:[x;y]\to\mathbb{R}$

$$f(t)=\arctan t \text{ and } g(t)=\ln t\overset{tc}{\Rightarrow}\exists\alpha\in(x;y)a\alpha$$

$$\frac{g'(\alpha)}{g'(x)}=\frac{\arctan x-\arctan y}{\ln x-\ln y}\text{ or }\frac{\frac{1}{1+\alpha^2}}{\frac{1}{\alpha}}=\frac{\arctan x-\arctan y}{\ln x-\ln y}$$

$$\frac{\arctan x-\arctan y}{\ln x-\ln y}=\frac{\alpha}{1+\alpha^2}$$

Consider $h:[1,\infty)\to\mathbb{R};h(\alpha)=\frac{\alpha}{1+\alpha^2}$

$$h'(\alpha)=\frac{1+\alpha^2-2\alpha^2}{(1+\alpha^2)^2}=\frac{1-\alpha^2}{(1+\alpha^2)^2}$$

α	1 $\qquad\qquad\qquad\qquad$ ∞
$h'(x)$	$-----------------$
$h(x)$	$\frac{1}{2}$

$h(1)=\frac{1}{2}$ *so,* $h(\alpha)\le\frac{1}{2}$ *then*

$$\int_1^a\int_1^b\frac{\arctan x-\arctan y}{\ln x-\ln y}dx\,dy=\int_1^a\int_1^b\frac{\alpha}{1+\alpha^2}dx\,dy<\int_1^a\int_1^b\frac{1}{2}dx\,dy=\frac{1}{2}(b-1)(a-1)\Rightarrow$$

$$\frac{2}{(b-1)(a-1)}\int_1^a\int_1^b\frac{\arctan x-\arctan y}{\ln x-\ln y}dx\,dy<1$$

764. Let p_n be the n^{th} prime number. Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{\sqrt[n]{p_n^3}+\sqrt[n]{p_n^5}-2}{3^{\sqrt[n]{p_n}}+5^{\sqrt[n]{p_n}}-8}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Mokhtar Khassani-Mostaganem-Algerie

$\Omega=\lim_{n\to\infty}\left(\frac{\sqrt[n]{P_n^3}+\sqrt[n]{P_n^5}-2}{3^{\sqrt[n]{P_n}}+5^{\sqrt[n]{P_n}}-8}\right)$. *Note that by prime number theorem:* $P_n\sim n\log n$

so: $\Omega=\lim_{n\to\infty}\left(\frac{\sqrt[n]{(n\log n)^3}+\sqrt[n]{(n\log n)^5}-2}{3^{\sqrt[n]{n\log n}}+5^{\sqrt[n]{n\log n}}-8}\right)$ $\qquad\left\{\lim_{n\to+\infty}\sqrt[n]{n\log n}=e^{\lim_{n\to+\infty}\frac{\log(n\log n)}{n}}=\right.$

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by L'Hopital rule we get:

$$\Omega=\lim_{n\to+\infty}\frac{\left(5\sqrt[n]{(n\log n)^3}+5\sqrt[n]{(n\log n)^5}\right)\left(\frac{1+\log n}{n^2\log n}-\frac{\log(n\log n)}{n^2}\right)}{\left(3^{\sqrt[n]{n\log n}}\log 3+5^{\sqrt[n]{n\log n}}\log 5\right)\left(\frac{1+\log n}{n^2\log n}-\frac{\log(n\log n)}{n^2}\right)}=\frac{8}{3\log 3+5\log 5}=\Omega$$

765. Prove that:

$$\lim_{n\to\infty} n\ln(n)\prod_{p\le n}\frac{p}{\sqrt[p-1]{p}(p+1)}=\frac{\pi^2}{6}$$

The product on the left-hand side extends over all prime numbers $p \le n$

Proposed by Prem Kumar-India

Solution by proposer

We are going to use Merten's 3rd theorem: $\lim_{n\to\infty}\frac{1}{\ln n}\prod_{p\le n}\frac{p}{p-1}=e^{\gamma}$

A well-known result: $\gamma=\lim_{n\to\infty}\left(\ln-\sum_{p<n}\frac{\ln p}{p-1}\right)\Rightarrow e^{\gamma}=\lim_{n\to\infty}\frac{n}{\sqrt[p-1]{p}}$

And Euler's product formula for $s=2$

$$\lim_{n\to\infty}\prod_{p\le n}\frac{p^2}{p^2-1}=\zeta(2)$$

$$LHS=\lim_{n\to\infty} n\ln n\prod_{p\le n}\frac{p}{\sqrt[p-1]{p}(p+1)}\times\frac{p}{p}\times\frac{p-1}{p-1}$$

$$=\lim_{n\to\infty}\prod_{p\le n}\frac{n}{\sqrt[p-1]{p}}\times\frac{1}{\frac{1}{\ln n}\times\frac{p}{p-1}}\times\frac{p^2}{p^2-1}=e^{\gamma}\times e^{-\gamma}\times\zeta(2)=\frac{\pi^2}{6}=RHS$$

Hence proved.

766. Find:

$$\Omega=\lim_{n\to\infty}\left(n\left(\sum_{k=1}^{n}\left(\log\left(\frac{2n+2k+1}{n+k}\right)\right)-n\log 2-\frac{\log 2}{2}\right)\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Remus Florin Stanca-Romania

$$\Omega=\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\left(\ln\left(\frac{2n+2k+1}{n+k}\right)\right)-n\ln 2-\frac{\ln 2}{2}}{\frac{1}{n}}\quad \textit{(b)}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\ln\left(\frac{2n+2k+1}{n+k}\right)\right)-n\ln 2-\frac{\ln 2}{2}=\lim_{n\to\infty}n\left(\ln\left(\sqrt[n]{\prod_{k=1}^{n}\left(\frac{2n+2k+1}{n+k}\right)}\right)-\ln 2\right)-\frac{\ln 2}{2}$$

(1)

$$\lim_{n\to\infty}\ln\left(\sqrt[n]{\prod_{k=1}^{n}\left(\frac{2n+2k+1}{n+k}\right)}\right)=\lim_{n\to\infty}\ln\left(\frac{\prod_{k=1}^{n+1}\frac{2n+2k+3}{n+k+1}}{\prod_{k=1}^{n}\frac{2n+2k+1}{n+k}}\right)=\lim_{n\to\infty}\ln\left(\frac{\frac{(4n+5)!!}{(2n+3)!!}\cdot\frac{(n+1)!}{(2n+2)!}}{\frac{(4n+1)!!}{(2n+1)!!}\cdot\frac{n!}{(2n)!}}\right)=$$

$$=\lim_{n\to\infty}\ln\left(\frac{(4n+3)}{(2n+3)}\cdot\frac{4n+5}{2n+1}\cdot\frac{n+1}{2n+2}\right)=\ln 2\Rightarrow n\left(\ln\left(\sqrt[n]{\prod_{k=1}^{n}\left(\frac{2n+2k+1}{n+k}\right)}\right)-\ln 2\right)$$

$$=\lim_{n\to\infty}\frac{\ln\left(\sqrt[n]{\prod_{k=1}^{n}\left(\frac{2n+2k+1}{n+k}\right)}\right)-\ln 2}{\frac{1}{n}}\overset{Stolz-Cesaro}{\underset{\frac{0}{0}}{=}}\lim_{n\to\infty}\frac{\ln\left(\frac{\sqrt[n+1]{\prod_{k=1}^{n+1}\left(\frac{2n+2k+3}{n+k+1}\right)}}{\sqrt[n]{\prod_{k=1}^{n}\left(\frac{2n+2k+1}{n+k}\right)}}\right)}{\frac{1}{n+1}-\frac{1}{n}}=$$

$$=\lim_{n\to\infty}\ln\left(\left(\frac{\prod_{k=1}^{n}\left(\frac{2n+2k+1}{n+k}\right)}{\prod_{k=1}^{n+1}\left(\frac{2n+2k+3}{n+k+1}\right)}\right)^{n}\cdot\prod_{k=1}^{n}\left(\frac{2n+2k+1}{n+k}\right)\right)=$$

$$=\lim_{n\to\infty}\ln\left(\prod_{k=1}^{n}\left(\frac{2n+2k+1}{2n+2k}\right)\right)\quad \textit{(a)}$$

$\frac{2n+2k+2}{2n+2k+1}<\frac{2n+2k+1}{2n+2k}<\frac{2n+2k}{2n+2k-1}$ ***is true because*** $(2n+2k)^2-1<(2n+2k)^2$ ***and***

$$(2n+2k)^2+2(2n+2k)<(2n+2k)^2+1+2(2n+2k)$$

$$\frac{2n+2k+2}{2n+2k+1}<\frac{2n+2k+1}{2n+2k}<\frac{2n+2k}{2n+2k-1}\,\Big|\cdot\frac{2n+2k+1}{2n+2k}\Rightarrow$$

$$\Rightarrow\sqrt{\frac{2n+2k+2}{2n+2k}}<\frac{2n+2k+1}{2n+2k}<\sqrt{\frac{2n+2k+1}{2n+2k-1}}\quad \textit{(2)}\Rightarrow$$

$$\Rightarrow\prod_{k=1}^{n}\sqrt{\frac{2n+2k+2}{2n+2k}}<\prod_{k=1}^{n}\frac{2n+2k+1}{2n+2k}<\prod_{k=1}^{n}\sqrt{\frac{2n+2k+1}{2n+2k-1}}$$

$$\prod_{k=1}^{n}\sqrt{\frac{2n+2k+2}{2n+2k}}=\sqrt{\frac{2n+4}{2n+2}\cdot\frac{2n+6}{2n+4}\cdot\ldots\cdot\frac{4n+2}{4n}}=\sqrt{\frac{4n+2}{2n+2}}\Rightarrow$$

$$\Rightarrow\lim_{n\to\infty}\prod_{k=1}^{n}\sqrt{\frac{2n+2k+2}{2n+2k}}=\sqrt{2}\quad \textit{(3)}$$

and in the same way $\lim_{n\to\infty}\prod_{k=1}^{n}\sqrt{\frac{2n+2k+1}{2n+2k-1}}=\sqrt{2}$ ***(4)***

$$\overset{(2);(3);(4)}{\Rightarrow}\lim_{n\to\infty}\prod_{k=1}^{n}\frac{2n+2k+1}{2n+2k}=\sqrt{2}\overset{(a)}{\Rightarrow}\lim_{n\to\infty}n\left(\ln\left(\sqrt[n]{\prod_{k=1}^{n}\left(\frac{2n+2k+1}{n+k}\right)}\right)-\ln 2\right)=$$

$$=\ln\left(\sqrt{2}\right)=\frac{\ln 2}{2}$$

$$\Rightarrow\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\left(\ln\left(\frac{2n+2k+1}{n+k}\right)\right)-n\ln 2-\frac{\ln 2}{2}}{\frac{1}{n}}\overset{\frac{0}{0}}{=}\lim_{n\to\infty}\frac{\ln\left(\prod_{k=1}^{n}\left(\frac{2n+2k+1}{n+k}\right)\cdot\frac{1}{2^n}\right)-\frac{\ln 2}{2}}{\frac{1}{n}}\underset{\frac{0}{0}}{\overset{Stolz-Cesaro}{=}}$$

$$=-\lim_{n\to\infty}\ln\left(\left(1+\frac{3}{16n^2+32n+12}\right)^{n^2+n}\right)=$$

$$=-\lim_{n\to\infty}\ln\left(\left(1+\frac{3}{16n^2+32n+12}\right)^{\frac{16n^2+32n+12}{3}\cdot\frac{3n^2+3n}{16n^2+32n+12}}\right)=-\ln\left(e^{\frac{3}{16}}\right)=-\frac{3}{16}\overset{(b)}{\Rightarrow}$$

$$\Rightarrow\Omega=-\frac{3}{16}$$

Solution 2 by Samir HajAli-Damascus-Syria

First: $\sum_{k=1}^{n}\ln\left(\frac{2n+2k+1}{n+k}\right)=\ln\left(\prod_{k=1}^{n}\frac{2\left(n+k+\frac{1}{2}\right)}{n+k}\right)=n\cdot\ln 2+\ln\left(\prod_{k=1}^{n}\frac{n+k+\frac{1}{2}}{n+k}\right)$

But: $\prod_{k=1}^{n}\left(n+k+\frac{1}{2}\right)=\frac{\Gamma\left(2n+\frac{3}{2}\right)}{\Gamma\left(n+\frac{3}{2}\right)}$

$$\prod_{k=1}^{n}(n+k)=\frac{\Gamma(2n+1)}{\Gamma(n+1)}$$

Then: $\sum_{k=1}^{n}\ln\left(\frac{2n+2k+1}{n+k}\right)=n\cdot\ln 2+\underbrace{\ln\left(\frac{\Gamma\left(2n+\frac{3}{2}\right)\times\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)\times\Gamma(2n+1)}\right)}_{=\theta_n}$

Therefore: $\Omega=\lim\left(n\left(\ln(\theta_n)+n\ln 2-n\ln 2-\frac{\ln 2}{2}\right)\right)$

$$\Omega=\lim\left(n\left(\ln(\theta_n)-\frac{\ln 2}{2}\right)\right)$$

But, by Laurent series: $\ln(\theta_n)=\frac{\ln 2}{2}-\frac{3}{16n}+\theta\left(\left(\frac{1}{n}\right)^{\frac{3}{2}}\right)$

$$\Leftrightarrow \Omega = \lim\left[n\left(\frac{\ln 2}{2} - \frac{3}{16n} - \frac{\ln 2}{2}\right)\right] = \lim_{n\to\infty}\left[-\frac{3}{16}\right] = -\frac{3}{16}$$

767. For any real number $m > 0$, let

$$\psi(m) = \int_0^\infty \frac{x}{(e^{\pi m x} - 1)} \cdot \frac{dx}{(1 + x^2)}$$

then evaluate the limit

$$\lim_{n\to\infty} \int_1^n \psi\left(m - \frac{1}{2}\right) dm$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Mokhtar Khassani-Mostaganem-Algerie

It is easy to prove that: $\Psi(z) = \log z - \frac{1}{2}z - 2\int_0^{+\infty} \frac{x}{(x^2+z^2)(e^{2\pi x}-1)} dx \quad |\arg z| < \frac{\pi}{2}$

so: $\psi(m) = \int_0^{+\infty} \frac{x}{(x^2+1)(e^{m\pi x}-1)} dx \ \left\{x \to \frac{2x}{m}\right\} = \int_0^{+\infty} \frac{x}{\left(x^2+\left(\frac{m}{2}\right)^2\right)(e^{2\pi x}-1)} dx =$

$$= \frac{1}{2}\left(\log\left(\frac{m}{2}\right) - \frac{1}{m} - \Psi\left(\frac{m}{2}\right)\right)$$

Now: $A(n) = \int_0^n \Psi\left(m - \frac{1}{2}\right) dm = \frac{1}{2}\int_0^n \left(\log\left(\frac{2m-1}{4}\right) - \frac{1}{2m-1} - \Psi\left(\frac{2m-1}{4}\right)\right) dm$

$$= \frac{1}{4}\left((2m-1)\log\left(\frac{2m-1}{4}\right) - 1 - 2\log(2m-1) + 2m - 4\log\left(\Gamma\left(\frac{2m-1}{4}\right)\right)\right)_1^n$$

$$= \left(\frac{1}{2}(2n-1)\log\left(\frac{2n-1}{4}\right) - 1 - \frac{1}{2}\log(2n-1) + \frac{1}{2}n - \log\left(\Gamma\left(\frac{2n-1}{4}\right)\right) - \frac{1 + \log\left(\frac{1}{4\Gamma^4\left(\frac{1}{4}\right)}\right)}{16}\right)$$

$$M = \lim A(n) = \frac{1}{4} + \log\left(\frac{1}{4\pi}\right) + 2\log\left(\Gamma\left(\frac{1}{4}\right)\right)$$

Note: $\log(\Gamma(a)) = \frac{2(a-1)\log a}{2} - a + \frac{1}{2}\log(2\pi) + 2\int_0^{+\infty} \frac{\arctan\left(\frac{z}{a}\right)}{e^{2\pi z}-1} dz$

768. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{n-1}{n^n}\sum_{k=1}^{n-1}k^k\right)$$

Proposed by Prem Kumar-India

Solution by Samir HajAli-Damascus-Syria

$$\Omega = \lim_{n\to\infty}\frac{n-1}{n^n}\sum_{k=1}^{n-1}k^k = \lim_{n\to\infty}\frac{n}{n^n}\sum_{k=1}^{n-1}k^k - \lim_{n\to\infty}\frac{1}{n^n}\sum_{k=1}^{n-1}k^k$$

$$= \lim_{n\to\infty}\frac{1}{n^{n-1}}\sum_{k=1}^{n-1}k^k - \lim_{n\to\infty}\frac{1}{n^n}\sum_{k=1}^{n-1}k^k = \Omega_1 - \Omega_2$$

Now: $\Omega_1 = \lim_{n\to\infty}\frac{1}{n^{n-1}}\sum_{k=1}^{n-1}k^k$. *Let put* $x_n = \sum_{k=1}^{n-1}k^k$ *then* $x_{n-1} = \sum_{k=1}^{n-2}k^k$

$y_n = n^{n-1}$ *then* $y_{n-1} = (n-1)^{n-2}$

so, $\lim_{n\to\infty}\frac{x_n-x_{n-1}}{y_n-y_{n-1}} = \lim_{n\to\infty}\frac{(n-1)^{n-1}}{n^{n-1}-(n-1)^{n-2}} = \lim_{n\to\infty}\frac{1}{\frac{n^{n-1}}{(n-1)^{n-1}}-\frac{1}{n-1}} = e^{-1}$

then by Stolz-Cesaro $\Omega_1 = e^{-1}$

and $\lim_{n\to\infty}\frac{1}{n^n}\sum_{k=1}^{n-1}k^k = \lim_{n\to\infty}\frac{(n-1)^{n-1}}{n^n-(n-1)^{n-1}} = \lim_{n\to\infty}\frac{1}{n\frac{n^{n-1}}{(n-1)^{n-1}}-1} = 0$

then $\Omega_2 = 0$. *Hence* $\Omega = e^{-1}$

769. Find:

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(\sum_{i=1}^{n}i(3i+1)\right)^{-1}\right)$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Sumit Kumar-India

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(\sum_{i=1}^{n}3i(i+1)\right)^{-1}\right)$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{3k(k+1)(2k+1)}{6}+\frac{k(k+1)}{2}\right)^{-1}$$

$$=\sum_{k=1}^{n}\left(\frac{k(k+1)(2k+1)}{2}+\frac{k(k+1)}{2}\right)^{-1}=\sum_{k=1}^{n}\left(\frac{1}{k(k+1)^2}\right)$$

$$=\sum_{k=1}^{n}\frac{1}{k(k+1)}-\frac{1}{(k+1)^2}=\sum_{k=1}^{n}\frac{1}{k(k+1)}-\sum_{1}^{n}\frac{1}{(k+1)^2}$$

$$=1-(\varphi(2)-1)=2-\varphi(2)$$

Solution 2 by Tobi Joshua-Nigeria

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(\sum_{i=1}^{n}i(3i+1)\right)^{-1}\right);\ \Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{\left(\sum_{i=1}^{k}i(3i+1)\right)}\right)$$

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{\left(\sum_{i=1}^{k}(3i^2)+\left(\sum_{i=1}^{k}i\right)\right)}\right);\ \Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{\frac{3k(k+1)(2k+1)}{6}+\frac{k(k+1)}{2}}\right)$$

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{2}{k(k+1)(2k+1+1)}\right);\ \Omega=\frac{1}{2}\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{2}{k(k+1)^2}\right)$$

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\frac{1}{k+1}\right)-\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{(k+1)^2}\right)$$

$$\Omega=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)-\left(\sum_{k=0}^{n}\frac{1}{(k+1)^2}-1\right);\ \Omega=\lim_{n\to\infty}\left(1-\frac{\frac{1}{n}}{\frac{1}{n}+1}\right)-\left(\sum_{k=1}^{\infty}\frac{1}{(k)^2}-1\right)$$

$$\Omega=1-(\zeta(2)-1);\ \Omega=2-\left(\frac{\pi^2}{6}\right)$$

Solution 3 by Yen Tung Chung-Taichung-Taiwan

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\sum_{i=1}^{n}i(3i+1)\right)^{-1}=\lim_{n\to\infty}\sum_{k=1}^{n}\left(3\sum_{i=1}^{n}i^2+\sum_{i=1}^{n}i\right)^{-1}$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\left(3\frac{k(k+1)(2k+1)}{6}+\frac{k(k+1)}{2}\right)^{-1} = \lim_{n\to\infty}\sum_{k=1}^{n}(k(k+1)^2)^{-1}$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)^2} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{(k+1)-k}{k(k+1)^2} = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k(k+1)}-\sum_{k=1}^{n}\frac{1}{(k+1)^2}\right)$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)-\sum_{k=2}^{\infty}\frac{1}{k^2} = \lim_{n\to\infty}\left\{\left(1-\frac{1}{n+1}\right)-\left(\sum_{k=1}^{\infty}\frac{1}{k^2}-1\right)\right\} = 2-\frac{\pi^2}{6}$$

770.

$$\Omega(p) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{k^{p-1}}{n^{p-1}\sqrt[p]{n^p+k^p}}\right), p\in\mathbb{N}, p\geq 1$$

Find:

$$\Omega = \lim_{p\to\infty}\Omega(p)$$

Proposed by Mohamed Rakkane-Casablanca-Morocco

Solution by Daniel Sitaru-Romania

$$\Delta_n = \left(0<\frac{1}{n}<\frac{2}{n}<\cdots<\frac{k-1}{n}<\frac{k}{n}<\cdots<\frac{n-1}{n}<1\right), \|\Delta_n\| = \frac{1}{n}\to 0$$

$$x_n^{k-1}\leq\xi_n^{k-1} = \frac{k-1}{n} < x_n^k, k\in\overline{1,n-1}, f:[0,1]\to\mathbb{R}, f(x) = \frac{x^{p-1}}{\sqrt[p]{1+x^p}}$$

$$\Omega(\mathrm{p}) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{k^{p-1}}{n^{p-1}\sqrt[p]{n^p+k^p}}\right) = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{p-1}\frac{1}{\sqrt[p]{1+\left(\frac{k}{n}\right)^p}}\right) =$$

$$= \lim_{n\to\infty}\left(\sum_{k=1}^{n}f\left(\frac{k}{n}\right)\left(\frac{k}{n}-\frac{k-1}{n}\right)\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}f(\xi_n^k)(x_n^k-x_n^{k-1})\right) = \lim_{n\to\infty}\sigma_{\Delta_n}(f,\xi_n) =$$

$$= \int_0^1 f(x)dx = \int_0^1\frac{x^{p-1}}{\sqrt[p]{1+x^p}}dx = \frac{1}{p}\int_0^1(1+x^p)'(1+x^p)^{-\frac{1}{p}}dx = \frac{1}{p}\left(2^{1-\frac{1}{p}}-1\right)$$

$$\Omega = \lim_{p\to\infty}\Omega(\mathrm{p}) = \lim_{p\to\infty}\frac{1}{p}\left(2^{1-\frac{1}{p}}-1\right) = 0$$

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771. Evaluate the following sum:

$$\sum_{k=1}^{\infty}\frac{1}{(3k)^3-3k}$$

Proposed by Prem Kumar-India

Solution 1 by Mokhtar Khassani-Mostaganem-Algerie

$$\sum_{k=1}^{+\infty}\frac{1}{(3k)^3-3k}=\sum_{k=1}^{+\infty}\frac{1}{3k(3k+1)(3k-1)}=\sum_{k=1}^{+\infty}\left(\frac{1}{2}\cdot\frac{1}{3k-1}+\frac{1}{2}\cdot\frac{1}{3k+1}-\frac{1}{3k}\right)=$$

$$=\frac{1}{6}\sum_{k=1}^{+\infty}\left(\frac{1}{k+\frac{1}{3}}-\frac{1}{k}\right)+\frac{1}{6}\sum_{k=1}^{+\infty}\left(\frac{1}{k-\frac{1}{3}}-\frac{1}{k}\right)=\frac{1}{6}\left(-\gamma-\Psi\left(\frac{4}{3}\right)\right)+\frac{1}{6}\left(-\gamma-\Psi\left(\frac{2}{3}\right)\right)=$$

$$=\frac{\log 3-1}{2}$$

Solution 2 by Avishek Mitra-West Bengal-India

$$\Leftrightarrow\underbrace{\phi(a,n)=1+2\sum_{k=1}^{n}\frac{1}{(ak)^3-ak}}_{Ramanujan's\ Phi\ function}\Rightarrow\underbrace{\phi(a)\equiv\lim_{n\to\infty}(\phi(a,n))=1+2\sum_{k=1}^{\infty}\frac{1}{(ak)^3-ak}}_{applying\ more\ properties\ of\ Ramanujan's\ phi\ function}$$

$$\underbrace{\Rightarrow 1+2\sum_{k=1}^{\infty}\frac{1}{(3k)^3-3k}=\varphi(3)=-\frac{1}{3}\left(\psi_0\left(\frac{1}{3}\right)+\psi_0\left(\frac{2}{3}\right)+2\gamma\right)}_{applying\ more\ properties\ of\ Ramanujan's\ phi\ function}$$

$$=\underbrace{-\frac{1}{3}\left(\psi_0\left(\frac{1}{3}\right)+\psi_0\left(\frac{1}{3}\right)+\pi\cot\left(\frac{\pi}{3}\right)+2\gamma\right)}_{applying\ digamma\ reflection\ equation}$$

$$=-\frac{1}{3}\left(2\psi_0\left(\frac{1}{3}\right)+\frac{\pi}{\sqrt{3}}+2\gamma\right)=-\frac{1}{3}\left(2\left(-\frac{\pi}{2\sqrt{3}}-\frac{3\ln 3}{2}-\gamma\right)+\frac{\pi}{\sqrt{3}}+2\gamma\right)$$

$$=-\frac{1}{3}\left(-\frac{\pi}{\sqrt{3}}-3\ln 3-2\gamma+\frac{\pi}{\sqrt{3}}+2\gamma\right)=\ln 3$$

$$\Leftrightarrow 1+2\sum_{k=1}^{\infty}\frac{1}{(3k)^3-3k}=\ln 3\Leftrightarrow\sum_{k=1}^{\infty}\frac{1}{(3k)^3-3k}=\frac{\ln 3-1}{2}$$

Solution 3 by Sumit Kumar-India

$$\sum_{k=1}^{\infty}\frac{1}{(3k)^2-3k}=\sum_{1}^{\infty}\frac{1}{(3k)(3k+1)(3k-1)}$$

$$=\sum_{1}^{\infty}\left(\frac{1}{3k-1}+\frac{1}{3k+1}-\frac{2}{3k}\right)\times\frac{1}{2}=\frac{1}{2}\sum_{1}^{\infty}\left(\frac{1}{3k-1}+\frac{1}{3k+1}-\frac{2}{3k}\right)$$

$$=\frac{1}{2}\sum_{1}^{\infty}\int_{0}^{1}x^{3k-2}+x^{3k}-2x^{3k-1}dx=\frac{1}{2}\int_{0}^{1}\sum_{k=1}^{\infty}x^{3k-2}+x^{3k}-2x^{3k-1}dx$$

$$=\frac{1}{2}\int_{0}^{1}\frac{x}{1-x^3}+\frac{x^3}{1-x^3}-\frac{2x^2}{1-x^3}dx=\frac{1}{2}\int_{0}^{1}\frac{x+x^3-2x^2}{1-x^3}dx$$

$$=\frac{1}{2}\int_{0}^{1}\frac{x(x^2-2x+1)}{(1-x)(1+x+x^2)}=\frac{1}{2}\int_{0}^{1}\frac{x(1-x)^2}{(1-x)(1+x+x^2)}dx$$

$$=\frac{1}{2}\int_{0}^{1}\frac{x-x^2}{1+x+x^2}dx=\frac{1}{2}\int_{0}^{1}\frac{2x+1}{1+x+x^2}-1dx=\frac{1}{2}\left[\ln(1+x+x^2)-x\right]_0^1=\frac{1}{2}(\ln 3-1)$$

Solution 4 by Samir HajAli-Damascus-Syria

$$S=\sum_{n=1}^{\infty}\frac{1}{(3n)^3-3n}$$

$$\frac{1}{(3n)^3-3n}=\frac{1}{27n^3-3n}=\frac{1}{3n}\times\frac{1}{9n^2-1}$$

$$=\frac{1}{3n}\times\frac{1}{(3n-1)(3n+1)}=\frac{1}{6n}\left(\frac{1}{3n-1}-\frac{1}{3n+1}\right)$$

$$\Rightarrow S=\sum_{n=1}^{\infty}\frac{1}{6n(3n-1)}-\sum_{n=1}^{\infty}\frac{1}{6n(3n+1)}$$

$$=\frac{1}{18}\left[\sum_{n=0}^{\infty}\frac{1}{(n+1)\left(n+\frac{2}{3}\right)}-\sum_{n=0}^{\infty}\frac{1}{(n+1)\left(n+\frac{4}{3}\right)}\right]=\frac{1}{18}\left[\frac{\psi(1)-\psi\left(\frac{2}{3}\right)}{1-\frac{2}{3}}-\frac{\psi\left(\frac{4}{3}\right)-\psi(1)}{\frac{4}{3}-1}\right]$$

$$=\frac{1}{6}\left[2\psi(1)-\psi\left(\frac{2}{3}\right)-\psi\left(\frac{4}{3}\right)\right]=\frac{1}{2}(\ln 3-1)$$

772. Evaluate the following sum:

$$\sum_{k=0}^{\infty}\frac{2k+1}{(2k+1)^4+64}$$

Proposed by Prem Kumar-India

Solution 1 by Mokhtar Khassani-Mostaganem-Algerie

$$\therefore\ (2k+1)^4+64=(2k+1)^4+8^2+16(2k+1)^2-16(2k+1)^2=$$

$$=((2k+1)^2+8)^2-16(2k+1)^2=$$

$$=\left((2k+1)^2+8-4(2k+1)\right)\left((2k+1)^2+8+4(2k+1)\right)=$$

$$=(4k^2-4k+5)(4k^2+12k+13)$$

$$\Omega=\sum_{k=0}^{+\infty}\frac{2k+1}{(2k+1)^4+64}=\frac{1}{8}\left[\sum_{k=0}^{+\infty}\frac{1}{4k^2-4k+5}-\frac{1}{4k^2+12k+13}\right]=$$

$$=\frac{1}{8}\left[\sum_{k=0}^{+\infty}\frac{1}{(2k-1)^2+5}-\frac{1}{(2k+3)^2+4}\right]=$$

$$=\frac{1}{8}\left[\frac{1}{5}+\sum_{k=0}^{+\infty}\left(\frac{1}{(2k-1)^2+4}-\frac{1}{(2k+1)^2+4}\right)+\frac{1}{5}\right]=\frac{1}{20}=\Omega$$

Solution 2 by Ahmed Bach-Bouira-Algerie

We have: $(2k+1)^4+64=0\Leftrightarrow k\in\left\{\frac{1}{2}+i;\frac{1}{2}-i;\ -\frac{3}{2}-i,-\frac{3}{2}+i\right\}$

Then: $(2k+1)^4+64=16\left(\left(k-\frac{1}{2}\right)^2+1\right)\left(\left(k+\frac{3}{2}\right)^2+1\right)$

$$\Rightarrow\frac{2k+1}{(2k+1)^4+64}=\frac{1}{32}\left(\frac{1}{\left(k-\frac{1}{2}\right)^2+1}-\frac{1}{\left(k+\frac{3}{2}\right)^2+1}\right)$$

$$=\frac{1}{8}\left(\frac{1}{(2k-1)^2+4}-\frac{1}{(2k+3)^2+4}\right)$$

So, we find:

$$\sum_{k=0}^{+\infty}\frac{2k+1}{(2k+1)^4+64}=\frac{1}{8}\times\frac{2}{5}=\frac{1}{20}$$

Solution 3 by Remus Florin Stanca-Romania

$$\sum_{k=0}^{\infty}\frac{2k+1}{(2k+1)^4+64}=\lim_{n\to\infty}\sum_{k=0}^{\infty}\frac{2k+1}{(2k+1)^4+64}=\lim_{n\to\infty}\sum_{k=0}^{\infty}\frac{2k+1}{((2k+1)^2+8)^2-16(2k+1)^2}$$

$$=\lim_{n\to\infty}\sum_{k=0}^{\infty}\frac{2k+1}{\left((2k+1)^2+8-4(2k+1)\right)\left((2k+1)^2+8+4(2k+1)\right)}=$$

$$=\lim_{n\to\infty}\sum_{k=0}^{\infty}\frac{2k+1}{(4k^2-4k+5)(4k^2+12k+13)}=$$

$$=\frac{1}{8}\cdot\lim_{n\to\infty}\frac{4k^2+12k+13-(4k^2-4k+5)}{(4k^2-4k+5)(4k^2+12k+13)}=$$

$$=\frac{1}{8}\lim_{n\to\infty}\sum_{k=0}^{\infty}\left(\frac{1}{4k^2-4k+5}-\frac{1}{4k^2+12k+13}\right)\quad \textbf{(1)}$$

Let $a_n=\frac{1}{4n^2-4n+5}\Rightarrow a_{n+2}=\frac{1}{4n^2+12n+13}\overset{(1)}{\Rightarrow}\lim_{n\to\infty}\sum_{k=0}^{\infty}\frac{2k+1}{(2k+1)^4+64}=\frac{1}{8}\lim_{n\to\infty}\sum_{k=0}^{\infty}(a_k-$

ak+2=

$$=\frac{1}{8}\lim_{n\to\infty}\left(a_0+\cdots+a_n-(a_2+\cdots+a_{n+2})\right)=\frac{1}{8}\lim_{n\to\infty}(a_0+a_1-a_{n+1}-a_{n+2})=$$

$$=\frac{1}{8}(a_0-a_1)=\frac{1}{8}\left(\frac{1}{5}+\frac{1}{5}\right)=\frac{1}{20}\Rightarrow\sum_{k=0}^{\infty}\frac{2k+1}{(2k+1)^4+64}=\frac{1}{20}$$

773. Find:

$$\Omega=\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\left(\int_{\varepsilon}^{1-\varepsilon}\left(\frac{\log(1-x)\cdot\log^3 x+\log^3 x}{1-x}\right)dx\right)$$

Proposed by Mohamed Arahman Jama-Somalia

Solution 1 by Mokhtar Khassani-Mostaganem-Algerie

$$\Omega=\int_0^1\frac{\log(1-x)\log^3 x+\log^3 x}{1-x}dx=M+N$$

$$N=\int_0^1\frac{\log^3 x}{1-x}dx=\int_0^1\sum_{n=0}^{+\infty}x^n\log^3 x\,dx=-3!\sum_{n=0}^{+\infty}\frac{1}{(n+1)^4}=-6\zeta(4)$$

$$M=\int_0^1\frac{\log(1-x)\log^3 x}{1-x}dx=-\int_0^1\sum_{n=1}^{+\infty}H_nx^n\log^3 x\,dx=3!\sum_{n=1}^{+\infty}\frac{H_n}{(n+1)^4}=$$

$$=6\big(2\zeta(5)-\zeta(2)\zeta(3)\big)$$

$$\Omega=6\big(2\zeta(5)-\zeta(2)\zeta(3)-\zeta(4)\big)$$

Solution 2 by Abdul Mukhtar-Nigeria

$$\Omega=\int_0^1\frac{\ln(1-x)\ln^3 x+\ln^3 x}{1-x}dx;\ \Omega=\int_0^1\frac{\ln(1-x)\ln^3 x}{1-x}dx+\int_0^1\frac{\ln^3 x}{1-x}dx$$

$$\Omega=I_1+I_2$$

$$I_1=\int_0^1\frac{\ln(1-x)\ln^3 x}{1-x}dx;\ I_1=-\sum_{n=1}^{\infty}H_n\int_0^1\ln^3 x\,x^n\,dx$$

Using $\int_0^1\ln^a(x)\,x^n\,dx=\frac{(-1)^a a!}{(n+1)^{a+1}}$ *where* $a=3$

$$I_1=\sum_{n=1}^{\infty}H_n\frac{3!}{(n+1)^4};\ I_1=6\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^4}=6\sum_{n=1}^{\infty}\frac{1}{(n+1)^4}\left[H_{n+1}-\frac{1}{n+1}\right]$$

$$I_1=6\sum_{n=1}^{\infty}\frac{H_{n+1}}{(n+1)^4}-6\sum_{n=1}^{\infty}\frac{1}{(n+1)^5};\ I_1=6\left[\sum_{n=1}^{\infty}\frac{H_n}{n^4}-\sum_{n=1}^{\infty}\frac{1}{n^5}\right]$$

$$I_1=6\big(3\zeta(5)-\zeta(2)\zeta(3)\big)-6\zeta(5);\ I_1=18\zeta(5)-6\zeta(2)\zeta(3)-6\zeta(5)$$

$$I_1=18\zeta(5)-6\zeta(5)-6\zeta(2)\zeta(3);\ I_1=18-6\big(\zeta(5)\big)-6\zeta(2)\zeta(3)$$

$I_1=12\zeta(5)-6\zeta(2)\zeta(3)$. *Now, for* $I_2=\int_0^1\frac{\ln^3 x}{1-x}dx$

$$I_2=\sum_{n=0}^{\infty}\int_0^1 x^{n-1}\ln^3 x\,dx;\ I_2=-6\sum_{n=1}^{\infty}\frac{1}{n^4}=-6\zeta(4)$$

$$\therefore\ \Omega=I_1+I_2$$

$$\Omega=12\zeta(5)-6\zeta(2)\zeta(3)+\big(-6\zeta(4)\big);$$

$$\Omega=12\zeta(5)-6\zeta(2)\zeta(3)-6\zeta(4)$$

$$\Omega=6[2\zeta(5)-\zeta(2)\zeta(3)-\zeta(4)]$$

774. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{n}\int_0^1 \left(\log\left(1 + e^{n\sin x}\right)\right)dx\right)$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Artan Ajredini-Presheva-Serbie

Integrating by parts we get:

$$\int_0^1 \log\left(1 + e^{n\sin x}\right)dx = \log\left(1 + e^{n\sin 1}\right) - n\int_0^1 \frac{x\cos x\, e^{n\sin x}}{1 + e^{n\sin x}}dx$$

Hence:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{n}\log\left(1 + e^{n\sin 1}\right) - \int_0^1 \frac{x\cos x\, e^{n\sin x}}{1 + e^{n\sin x}}dx\right) = \sin 1 - \int_0^1 x\cos x\, dx =$$

$$= \sin 1 - \sin 1 - \cos 1 + 1 = 1 - \cos 1$$

Solution 2 by Mokhtar Khassani-Mostaganem-Algerie

$$\lim_{n\to+\infty}\frac{1}{n}\int_0^1 \log\left(1 + e^{n\sin x}\right)dx\ \{Beppo\ levi\} = \int_0^1 \lim_{n\to+\infty}\frac{\log\left(1 + e^{n\sin x}\right)}{n}dx =$$

$$= \int_0^1 \lim_{n\to+\infty}\frac{n\sin x + \log\left(1 + \frac{1}{e^{n\sin x}}\right)}{n}dx = \int_0^1 \sin x\, dx = 1 - \cos 1$$

775. Find:

$$\sum_{n=0}^{\infty}\frac{n + 2}{2^n(n + 1)^2}\left(H_{n+1}\right)$$

Proposed by Abdul Mukhtar-Nigeria

Solution by Mokhtar Khassani-Mostaganem-Algerie

$$M=\sum_{n=0}^{+\infty}\frac{n+2}{2^n(n+1)^2}H_{n+1}=2\sum_{n=1}^{+\infty}\frac{H_n}{2^n n}+\sum_{n=1}^{+\infty}\frac{H_n}{2^n n^2}=\zeta(2)(1-\log 2)+2\zeta(3)$$

Note: $\frac{\log(1-x)}{x-1}=\sum_{n=0}^{+\infty}H_n x^n\Rightarrow\sum_{n=0}^{+\infty}\frac{H_n}{n}x^n=\int\frac{\log(1-x)}{x(x-1)}dx=C+\frac{\log^2(1-y)}{2}+Li_2(x)\;\; C=0$ ***here***

$$\sum_{n=0}^{+\infty}\frac{H_n}{n^2}x^n=\int\frac{\log^2(1-x)}{2x}+\int\frac{Li_2(x)}{x}dx=$$

$$=Li_3(x)+\frac{\log^2(1-x)\log x}{2}+\int\frac{\log(1-x)\log x}{1-x}dx$$

$$=Li_3(x)+\frac{\log^2(1-x)\log x}{2}+\log(1-x)\,Li_2(1-x)-\int\frac{Li_2(1-x)}{1-x}dx=$$

$$=C+Li_3(x)+\frac{\log^2(1-x)\log x}{2}+\log(1-x)\,Li_2(1-x)-Li_3(1-x)$$

$C=\zeta(3)$ ***here*** $\therefore$ ***so just applied*** $x=\frac{1}{2}$

776. **Find:**

$$\Omega=\lim_{n\to\infty}\int_{e^{H_n}}^{e^{H_{n+1}}}\left(\tan^{-1}\left(\frac{x}{n}\right)\right)^5dx$$

Proposed by Daniel Sitaru – Romania

Solution by Remus Florin Stanca – Romania

Let $f(x)=\left(\tan^{-1}\left(\frac{x}{n}\right)\right)^5$, f ***is continuous on*** $I=[e^{H_n};e^{H_{n+1}}]\Rightarrow\exists\zeta\in I$ ***such that***

$$\int_{e^{H_n}}^{e^{H_{n+1}}}\left(\tan^{-1}\left(\frac{x}{n}\right)\right)^5=(e^{H_{n+1}}-e^{H_n})\cdot\left(\tan^{-1}\left(\frac{\zeta}{n}\right)\right)^5\Rightarrow\lim_{n\to\infty}\int_{e^{H_n}}^{e^{H_{n+1}}}\left(\tan^{-1}\left(\frac{x}{n}\right)\right)^5dx=$$

$$=\lim_{n\to\infty}(e^{H_{n+1}}-e^{H_n})\left(\tan^{-1}\left(\frac{\zeta}{n}\right)\right)^5=$$

$$=\lim_{n\to\infty}e^{H_n}(e^{H_{n+1}-H_n}-1)\left(\tan^{-1}\left(\frac{\zeta}{n}\right)\right)^5=\lim_{n\to\infty}e^{H_n-\ln n}\cdot\frac{e^{\frac{1}{n+1}}-1}{\frac{1}{n+1}}\cdot\left(\tan^{-1}\left(\frac{\zeta}{n}\right)\right)^5=$$

$$= e^{\gamma}\cdot\lim_{n\to\infty}\left(\tan^{-1}\left(\frac{\zeta}{n}\right)^5\right)\ \textit{(1)}$$

$$e^{H_n}\le\zeta\le e^{H_{n+1}}\Rightarrow\frac{e^{H_n}}{n}\le\frac{\zeta}{n}\le\frac{e^{H_{n+1}}}{n}\ \textit{(2)}$$

$$\lim_{n\to\infty}\frac{e^{H_n}}{n}=\lim_{n\to\infty}e^{H_n-\ln n}=e^{\gamma}\ \textit{(3)}$$

$$\lim_{n\to\infty}\frac{e^{H_{n+1}}}{n}=\lim_{n\to\infty}e^{H_{n+1}-\ln(n+1)}=e^{\gamma}\ \textit{(4)}$$

$$f\text{ is continuous}\overset{(2);(3);(4)}{\Rightarrow}\lim_{n\to\infty}\frac{\zeta}{n}=e^{\gamma}\overset{(1)}{\Rightarrow}e^{\gamma}\cdot(\tan^{-1}(e^{\gamma}))^5$$

777. Find:

$$\Omega=\sum_{k=1}^{\infty}\left(\frac{1}{1^4+2^4+\dots+k^4}\right)$$

Proposed by Vasile Mircea Popa – Romania

Solution by Mokhtar Khassani-Mostaganem-Algerie

$$\Omega=\sum_{k=1}^{+\infty}\frac{1}{1^4+2^4+\dots+k^4}=\sum_{k=1}^{+\infty}\frac{30}{n(n+1)(2n+1)(3n^2+3n-1)}$$

$$=\sum_{k=1}^{+\infty}\frac{5}{n(n+1)\left(n+\frac{1}{2}\right)\left(n+\frac{3+\sqrt{21}}{6}\right)\left(n+\frac{3-\sqrt{21}}{6}\right)}=$$

$$=\sum_{k=1}^{+\infty}\left(\frac{90}{7\left(n+\frac{3+\sqrt{21}}{6}\right)}+\frac{90}{7\left(n+\frac{3-\sqrt{21}}{6}\right)}+\frac{240}{7\left(n+\frac{1}{2}\right)}-\frac{30}{n+1}-\frac{30}{n}\right)$$

$$=\sum_{k=1}^{+\infty}\left(\frac{90}{7}\left(\frac{1}{n+\frac{3-\sqrt{21}}{6}}-1\right)+\frac{90}{7}\left(\frac{1}{n+\frac{3+\sqrt{21}}{6}}-\frac{1}{n}\right)+\frac{240}{7}\left(\frac{1}{n+\frac{1}{2}}-\frac{1}{n}\right)-30\left(\frac{1}{n+1}-\frac{1}{n}\right)\right)$$

$$=-\frac{90}{7}\left(\Psi\left(\frac{9-\sqrt{21}}{6}\right)+\gamma\right)-\frac{90}{7}\left(\Psi\left(\frac{9+\sqrt{21}}{6}\right)+\gamma\right)-\frac{240}{7}\left(\Psi\left(\frac{3}{2}\right)+\gamma\right)+30$$

$$\therefore \Omega = \frac{30}{7}\left(7 - 14\gamma - 8\Psi\left(\frac{3}{2}\right) - 3\Psi\left(\frac{9-\sqrt{21}}{6}\right) - 3\Psi\left(\frac{9+\sqrt{21}}{6}\right)\right)$$

Note: $\sum_{k=1}^{+\infty}\left(\frac{1}{n} - \frac{1}{n+z}\right) = \gamma + \Psi(1+z) \therefore \Psi$: ***the Digamma function.***

778. $\pi(x)$ **– number of prime numbers less than** $x, x \geq 2$**. Find:**

$$\Omega = \lim_{x\to\infty}\left(\frac{\pi(1+\log(1+3x))}{\pi(1+\log(1+2x))}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Adrian Popa-Romania

$$\lim_{x\to\infty}\frac{\pi(x)}{\frac{x}{\ln x}} = 1$$

$$\Omega = \lim_{x\to\infty}\left(\frac{\pi(1+\ln(1+3x))}{\pi(1+\ln(1+2x))}\right) = \lim_{x\to\infty}\frac{\frac{1+\ln(1+3x)}{\ln(1+\ln(1+3x))}}{\frac{1+\ln(1+2x)}{\ln(1+\ln(1+2x))}} =$$

$$= \lim_{x\to\infty}\frac{\dfrac{1+\frac{\ln(1+3x)}{3x}\cdot 3x}{\frac{\ln(1+\ln(1+3x))}{\ln(1+3x)}\cdot\ln(1+3x)}}{\dfrac{1+\frac{\ln(1+2x)}{2x}\cdot 2x}{\frac{\ln(1+\ln(1+2x))}{\ln(1+2x)}\cdot\ln(1+2x)}} = \lim_{x\to\infty}\frac{\frac{1+3x}{\ln(1+3x)}}{\frac{1+2x}{\ln(1+2x)}} =$$

$$= \lim_{x\to\infty}\frac{1+3x}{\frac{\ln(1+3x)}{3x}\cdot 3x}\cdot\frac{\frac{\ln(1+2x)}{2x}\cdot 2x}{1+2x} = \lim_{x\to\infty}\frac{1+3x}{3x}\cdot\frac{2x}{1+2x} = \lim_{x\to\infty}\frac{6x+2}{6x+3} = 1$$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

As you know: $\lim_{x\to\infty}\left(\frac{\pi(x)}{\left(\frac{x}{\ln(x)}\right)}\right) = 1$, ***(by using the Theorem of A. Sellerg and P. Erodes)***

$$\Omega = \lim_{x\to\infty}\left[\frac{\frac{1+\log(1+3x)}{\log(1+\log(1+3x))}\cdot\left(\frac{\frac{\pi(1+\log(1+3x))}{1+\log(1+3x)}}{\log(1+\log(1+3x))}\right)}{\frac{1+\log(1+2x)}{\log(1+\log(1+2x))}\cdot\left[\frac{\frac{\pi(1+\log(1+2x))}{1+\log(1+2x)}}{\log(1+\log(1+2x))}\right]}\right]$$

Suppose: $g(x) = \frac{1+\log(1+3x)}{\log(1+\log(1+3x))}\cdot\frac{\log(1+\log(1+2x))}{1+\log(1+2x)}$

$$g(x) = \left(\frac{1+\log(1+3x)}{1+\log(1+2x)}\right)\left(\frac{\log(1+\log(1+2x))}{\log(1+\log(1+3x))}\right)$$

$$h(x) = \frac{1+\log(1+3x)}{1+\log(1+2x)}, \lim_{x\to\infty}(h(x)) = \lim_{x\to\infty}\frac{\left(\frac{3}{1+3x}\right)}{\left(\frac{2}{1+2x}\right)} = \lim_{x\to\infty}\frac{3}{1+3x}\cdot\frac{1+2x}{2} =$$

$$= \lim_{x\to\infty}\frac{1+2x}{1+3x}\cdot\frac{3}{2} = \frac{2}{3}\cdot\frac{3}{2} = 1$$

$$k(x) = \frac{\log(1+\log(1+2x))}{\log(1+\log(1+3x))}, \lim_{x\to\infty}(k(x)) = \lim_{x\to 0}\left[\frac{\frac{\frac{2}{1+2x}}{}}{\frac{\frac{3}{1+3x}}{1+\log(1+3x)}}\right]$$

$$\lim_{x\to\infty}[k(x)] = \lim_{x\to\infty}\left[\frac{2}{1+2x}\cdot\frac{1}{1+\log(1+2x)}\cdot\frac{(1+3x)}{3}\cdot\frac{1+\log(1+3x)}{1}\right]$$

$= \lim_{x\to\infty}\left[\frac{2}{3}\cdot\frac{1+3x}{1+2x}\cdot\frac{1+\log(1+3x)}{1+\log(1+2x)}\right] = \frac{2}{3}\cdot\frac{3}{2}\cdot 1 = 1$. ***So:*** $\Omega = \lim_{x\to+\infty}\left(\frac{\pi(1+\log(1+3x))}{\pi(1+\log(1+2x))}\right) = 1\cdot$

$$1 = 1$$

Solution 3 by Jalil Hajimir-Canada

$$\Omega = \lim_{x\to\infty}\frac{1+\log(1+3x)}{\log(1+\log(1+3x))}\cdot\frac{\log(1+\log(1+2x))}{1+\log(1+2x)} = 1$$

Solution 4 by Remus Florin Stanca-Romania

We know that $\lim_{x\to\infty}\frac{\pi(x)\cdot\ln(x)}{x} = 1$

$$\Omega = \lim_{x\to\infty}\frac{\pi(1+\ln(1+3x))\cdot\ln(1+\ln(1+3x))}{1+\ln(1+3x)}\cdot$$

$$\cdot\frac{1+\ln(1+2x)}{\pi(1+\ln(1+2x))\cdot\ln(1+\ln(1+2x))}\cdot\frac{\ln(1+\ln(1+2x))}{1+\ln(1+2x)}\cdot\frac{1+\ln(1+3x)}{\ln(1+\ln(1+3x))}=$$

$$=\lim_{x\to\infty}\frac{\ln(1+\ln(1+2x))}{\ln(1+\ln(1+3x))}\cdot\frac{1+\ln(1+3x)}{1+\ln(1+2x)}\overset{L'H}{\underset{\frac{\infty}{\infty}}{=}}\lim_{x\to\infty}\frac{\frac{1}{1+\ln(1+2x)}\cdot\frac{2}{1+2x}}{\frac{1}{1+\ln(1+3x)}\cdot\frac{3}{1+3x}}\cdot\lim_{x\to\infty}\frac{\frac{3}{1+3x}}{\frac{2}{1+2x}}=$$

$$=\lim_{x\to\infty}\frac{\frac{1}{1+\ln(1+2x)}}{\frac{1}{1+\ln(1+3x)}}\cdot\lim_{x\to\infty}1\overset{L'H}{\underset{\frac{\infty}{\infty}}{=}}\lim_{x\to\infty}\frac{\frac{3}{1+3x}}{\frac{2}{1+2x}}=1\Rightarrow\Omega=1$$

779. Find:

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(\sum_{i=1}^{k}(4i-1)\right)^{-1}\right)$$

Proposed by Vasile Mircea Popa-Romania

Solution by Samir HajAli-Damascus-Syria

$$\sum_{i=1}^{k}(4i-1)=\sum_{i=1}^{k}(4i-4+3)=4\cdot\sum_{i=1}^{k}(i-1)+3\sum_{i=1}^{k}1$$

$$=4\times\frac{k(k-1)}{2}+3k=2k^2-2k+3k=2k^2+k$$

$$=2\left(k^2+\frac{1}{2}k+\frac{1}{16}-\frac{1}{16}\right)=2\left(k+\frac{1}{4}\right)^2-\frac{1}{8}\Rightarrow\left(\sum_{i=1}^{k}(4i-1)\right)^{-1}=\left(2\left(k+\frac{1}{4}\right)^2-\frac{1}{8}\right)^{-1}$$

$$=\left(\frac{1}{8}(4k+1)^2-\frac{1}{8}\right)^{-1}=8((4k+1)^2-1)^{-1}=\frac{8}{(4k+1)^2-1}$$

$$=\frac{8}{(4k+1-1)(4k+1+1)}=\frac{8}{(4k)(4k+2)}=\frac{1}{2}\times\frac{1}{k\left(k+\frac{1}{2}\right)}$$

Now: $\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{i=1}^{k}(4i-1)^{-1}=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k\left(k+\frac{1}{2}\right)}$

$$=\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(k+1)\left(k+\frac{3}{2}\right)}=\frac{1}{2}\left(\frac{\psi\left(\frac{3}{2}\right)-\psi(1)}{\frac{3}{2}-1}\right)=2-\ln(4)$$

780. Find:

$$\Omega(p)=\lim_{p\to\infty}\left(\lim_{n\to\infty}\left(\prod_{k=1}^{n}\left(1+\frac{(1+k)^p}{n^{p+1}}\right)\right)\right)^p, p>0$$

Proposed by Florică Anastase-Romania

Solution by Mokhtar Khassani-Mostaganem-Algerie

Let: $M=\lim_{n\to+\infty}\left(\prod_{k=1}^{n}\left(1+\frac{(k+1)^p}{n^{p+1}}\right)\right)=\lim_{n\to+\infty}\left(\frac{1+\frac{(n+1)^p}{n^{p+1}}}{1+\frac{1}{n^{p+1}}}\prod_{k=1}^{n}\left(1+\frac{k^p}{n^{p+1}}\right)\right)=$

$$=\lim_{n\to+\infty}\left(\prod_{k=1}^{n}\left(1+\frac{k^p}{n^{p+1}}\right)\right)=\exp\left(\lim_{n\to+\infty}\sum_{k=1}^{n}\log\left(1+\frac{k^p}{n^{p+1}}\right)\right)$$

$$\therefore\sum_{k=1}^{n}\ln\left(1+\frac{(k+1)^p}{n^{p+1}}\right)=\sum_{k=1}^{n}\frac{\log\left(1+\frac{k^p}{n^{p+1}}\right)}{\frac{k^p}{n^{p+1}}}\cdot\frac{k^p}{n^{p+1}}$$

$$\Rightarrow\lim_{n\to+\infty}\left\{\inf_{k=\overline{1,n}}\left[\frac{\log\left(1+\frac{k^p}{n^{p+1}}\right)}{\frac{k^p}{n^{p+1}}}\right]\sum_{k=1}^{n}\frac{k^p}{n^{p+1}}\right\}<\lim_{n\to+\infty}\sum_{k=1}^{n}\ln\left(1+\frac{k^p}{n^{p+1}}\right)<$$

$$<\lim_{n\to+\infty}\left\{\sup_{k=\overline{1,n}}\left[\frac{\log\left(1+\frac{k^p}{n^{p+1}}\right)}{\frac{k^p}{n^{p+1}}}\right]\sum_{k=1}^{n}\frac{k^p}{n^{p+1}}\right\}$$

$$\Rightarrow\frac{1}{p+1}=\int_0^1 x^p\,dx=\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^p<\lim_{n\to+\infty}\sum_{k=1}^{n}\ln\left(1+\frac{k^p}{n^{p+1}}\right)<$$

$$<\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^p=\int_0^1 x^p\,dx=\frac{1}{p+1}$$

So: $M=e^{\left(\lim_{n\to+\infty}\sum_{k=1}^{n}\ln\left(1+\frac{k^p}{n^{p+1}}\right)\right)}=e^{\frac{1}{p+1}}$

Now: $\Omega(p) = \lim_{p\to+\infty} M^P = \lim_{p\to+\infty} e^{\frac{p}{p+1}} \therefore \Omega(p) = e$

781.

$$\Omega_n + H_n = \sum_{k=3}^{n}\left(\left(\frac{\sqrt[k+1]{(k+1)!}}{\sqrt[k]{k!}}\right)^{k+1} \cdot \left(\frac{\sqrt[k-1]{(k-1)!}}{\sqrt[k]{k!}}\right)^{k-1}\right), n \in \mathbb{N}, n \geq 3$$

Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\Omega_n}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ali Jaffal-Lebanon

$$\Omega_n + H_n = \sum_{k=3}^{k=n}\left(\left(\frac{\sqrt[k+1]{(k+1)!}}{\sqrt[k]{k!}}\right)^{k+1} \left(\frac{\sqrt[k-1]{(k-1)!}}{\sqrt[k]{k!}}\right)^{k-1}\right)$$

$$= \sum_{k=3}^{k=n} \frac{(k+1)!}{(k!)^{1+\frac{1}{k}}} \cdot \frac{(k-1)!}{(k!)^{1-\frac{1}{k}}} = \sum_{k=3}^{k=n} \frac{(k+1)! \cdot (k-1)!}{k! \cdot k!} = \sum_{k=3}^{k=n} \frac{(k+1)!}{k!} \cdot \frac{(k-1)!}{k!} = \sum_{k=3}^{k=n} (k+1) \times \frac{1}{k}$$

$$\Omega_n + H_n = \sum_{k=3}^{k=n} 1 + \sum_{k=3}^{k=n} \frac{1}{k} = (n-2) + H_n - 1 - \frac{1}{2}$$

$$\Omega_n = n - 2 - \frac{3}{2} = n - \frac{7}{2}, \text{ so, } \lim_{n\to\infty} \sqrt[n]{\Omega_n} = 1$$

Solution 2 by Remus Florin Stanca-Romania

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\Omega_n} = \lim_{n\to\infty} \frac{\Omega_{n+1}}{\Omega_n} =$$

$$= \lim_{n\to\infty} \frac{\sum_{k=3}^{n+1}\left(\left(\frac{\sqrt[k+1]{(k+1)!}}{\sqrt[k]{k!}}\right)^{k+1} \cdot \left(\frac{\sqrt[k-1]{(k-1)!}}{\sqrt[k]{k!}}\right)^{k-1}\right) - H_{n+1}}{\sum_{k=3}^{n}\left(\left(\frac{\sqrt[k+1]{(k+1)!}}{\sqrt[k]{k!}}\right)^{k+1} \cdot \left(\frac{\sqrt[k-1]{(k-1)!}}{\sqrt[k]{k!}}\right)^{k-1}\right) - H_n}$$

$$\overset{Stolz\ Cesaro}{=} \lim_{n\to\infty} \frac{\left(\frac{\sqrt[n+3]{(n+3)!}}{\sqrt[n]{(n+2)!}}\right)^{n+3} \cdot \left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n+2]{(n+2)!}}\right)^{n+1} - \frac{1}{n+2}}{\left(\frac{\sqrt[n+2]{(n+2)!}}{\sqrt[n+1]{(n+1)!}}\right)^{n+2} \cdot \left(\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}\right)^{n} - \frac{1}{n+1}} \quad (1)$$

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+3]{(n+3)!}}{\sqrt[n+2]{(n+2)!}}\right)^{n+3} = \lim_{n\to\infty}\frac{(n+3)!}{(n+2)!\sqrt[n+2]{(n+2)!}} = \lim_{n\to\infty}\frac{n+3}{\sqrt[n+2]{(n+2)!}} =$$

$$= \lim_{n\to\infty}\sqrt[n+2]{\frac{(n+3)^{n+2}}{(n+2)!}} = \lim_{n\to\infty}\frac{(n+4)^{n+4}}{(n+3)!}\cdot\frac{(n+2)!}{(n+3)^{n+2}} =$$

$$= \lim_{n\to\infty}\frac{1}{n+3}\cdot\frac{(n+4)^{n+3}}{(n+3)^{n+2}} = \lim_{n\to\infty}\left(1+\frac{1}{n+3}\right)^{n+3} = e(2) \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{\sqrt[n+2]{(n+2)!}}{\sqrt[n+1]{(n+1)!}}\right)^{n+2} = \lim_{n\to\infty}\frac{\sqrt[n+2]{(n+2)!}}{\sqrt[n+1]{(n+1)!}}\cdot\lim_{n\to\infty}\left(\frac{\sqrt[n+2]{(n+2)!}}{\sqrt[n+1]{(n+1)!}}\right)^{n+1} = e \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty}\frac{\frac{\sqrt[n+2]{(n+2)!}}{n+2}}{\frac{\sqrt[n+1]{(n+1)!}}{n+1}}\cdot\lim_{n\to\infty}\left(\frac{\sqrt[n+2]{(n+2)!}}{\sqrt[n+1]{(n+1)!}}\right)^{n+1} = e \Rightarrow \lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n+2]{(n+2)!}}\right)^{n+1} = \frac{1}{e} \quad (3)$$

$$\overset{(2);(3)}{\Rightarrow} \lim_{n\to\infty}\left(\frac{\sqrt[n+3]{(n+3)!}}{\sqrt[n+2]{(n+2)!}}\right)^{n+3}\cdot\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n+2]{(n+2)!}}\right)^{n+1} = 1 \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{\sqrt[n+3]{(n+3)!}}{\sqrt[n+2]{(n+2)!}}\right)^{n+3}\cdot\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n+2]{(n+2)!}}\right)^{n+1} - \frac{1}{n+2} = 1 \Leftrightarrow$$

$$\Leftrightarrow \lim_{n\to\infty}\left(\frac{\sqrt[n+2]{(n+2)!}}{\sqrt[n+1]{(n+1)!}}\right)^{n+2}\cdot\left(\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}\right)^{n} - \frac{1}{n+1} = 1 \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty}\frac{\left(\frac{\sqrt[n+3]{(n+3)!}}{\sqrt[n+2]{(n+2)!}}\right)^{n+3}\cdot\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n+2]{(n+2)!}}\right)^{n+1} - \frac{1}{n+2}}{\left(\frac{\sqrt[n+2]{(n+2)!}}{\sqrt[n+1]{(n+1)!}}\right)^{n+2}\cdot\left(\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}\right)^{n} - \frac{1}{n+1}} = 1 \overset{(1)}{\Rightarrow} \lim_{n\to\infty}\sqrt[n]{\Omega_n} = 1 \Rightarrow \Omega = 1$$

Solution 3 by Tobi Joshua-Nigeria

$$\Omega_n + H_n = \sum_{k=3}^{n}\left(\left(\frac{\sqrt[k+1]{(k+1)!}}{\sqrt[k]{k!}}\right)^{k+1}\cdot\left(\frac{\sqrt[k-1]{(k-1)!}}{\sqrt[k]{k!}}\right)^{k-1}\right), n\in\mathbb{N}, n\geq 3$$

$$\Omega_n + H_n = \sum_{k=3}^{n}\left(\left(\frac{(k+1)!}{(k!)^{\frac{k+1}{k}}}\right)\cdot\left(\frac{(k-1)!}{(k!)^{\frac{k-1}{k}}}\right)\right) \textit{since } H_n = \sum_{k=1}^{n}\left(\frac{1}{k}\right) \quad \ominus$$

$$\Omega_n = \sum_{k=3}^{n}\left(\left(\frac{(k+1)!\,(k-1)!}{(k!)^2}\right)\right) - \sum_{k=1}^{n}\left(\frac{1}{k}\right);\ \Omega_n = \sum_{k=0}^{n}\left(\frac{(k+4)!\,(k+2)!}{(k+3)!^2}\right) - \sum_{k=0}^{n}\left(\frac{1}{k+1}\right)$$

$$\Omega_n = \sum_{k=0}^{n}\left(\frac{(k+4)}{(k+3)}\right) - \sum_{k=0}^{n}\left(\frac{1}{k+1}\right);\ \Omega_n = \sum_{k=0}^{n}\left(\frac{(k+4)(k+1)-(k+3)}{(k+3)(k+1)}\right)$$

Applying telescoping series

$$\Omega_n = \sum_{k=0}^{n}\left(\frac{k^2+4k+1}{(k+3)(k+1)}\right) = \frac{1}{(n+2)} + \frac{1}{(n+3)} - \frac{1}{2} + n$$

$$\textit{Then} \Leftrightarrow \sqrt[n]{\Omega_n} = \left(\frac{1}{(n+2)} + \frac{1}{(n+3)} - \frac{1}{2} + n\right)^{\frac{1}{n}}$$

$$\textit{then } \lim_{n\to\infty}\sqrt[n]{\Omega_n} = \lim_{n\to\infty} e^{\frac{1}{n}\log\left(\frac{1}{(n+2)}+\frac{1}{(n+3)}-\frac{1}{2}+n\right)} = e^0 = 1$$

$$\lim_{n\to\infty}\sqrt[n]{\Omega_n} = 1$$

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$$\Omega = \lim_{n\to\infty}\left(a^{\frac{1}{n+1}} + a^{\frac{1}{n+2}} + \cdots + a^{\frac{1}{2n}} - n\right)$$

Prove that:

$$\frac{1}{\log 2}\left(a\Omega(b) + b\Omega(a)\right) > \log\left(\frac{ab}{(a+b)^2}\right), a, b \in (0,1)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Khaled Abd Imouti-Damascus-Syria

$$u_n = \underbrace{a^{\frac{1}{n+1}} + a^{\frac{1}{n+2}} + \cdots + a^{\frac{1}{2n}}}_{t_n} - n;\ n\cdot a^{\frac{1}{2n}} \le t_n \le n\cdot a^{\frac{1}{n+1}}$$

$$n\cdot a^{\frac{1}{2n}} - n \le t_n - n \le n\cdot a^{\frac{1}{n+1}} - n;\ n\left(a^{\frac{1}{2n}} - 1\right) \le u_n \le n\left(a^{\frac{1}{n+1}} - 1\right)$$

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$$\frac{a^{\frac{1}{2n}}-1}{\frac{1}{n}} \le u_n \le \frac{a^{\frac{1}{n+1}}-1}{\frac{1}{n}};\ \frac{e^{\frac{1}{2n}\ln(a)}-1}{\frac{1}{n}} \le u_n \le \frac{e^{\frac{1}{n+1}\ln(a)}-1}{\frac{1}{n}}$$

Suppose:

$$f(x)=\frac{e^{\frac{1}{2n}\ln(a)}-1}{\frac{1}{n}},\ \lim_{n\to+\infty} f(x)=\lim_{n\to+\infty}\left[\frac{\frac{-\ln(a)}{2x^2}e^{\frac{1}{2x}\ln(a)}}{-\frac{1}{x^2}}\right]=\frac{1}{2}\ln(a)$$

Suppose: $g(x)=\frac{e^{\frac{1}{n+1}\ln(a)}-1}{\frac{1}{n}}$, $\lim_{n\to+\infty} g(x)=\lim_{n\to+\infty}\left[\frac{\frac{-\ln(a)}{(x+1)^2}\cdot e^{\frac{1}{n+1}\ln(a)}}{-\frac{1}{n^2}}\right]=\ln(a)$

So: $\frac{1}{2}\ln(a) \le \lim_{n\to+\infty} u_n \le \ln(a)$; $\frac{1}{2}\ln(a) \le \Omega(a) \le \ln(a)$

$$\frac{1}{\log 2}\left(a\cdot\Omega(b)+b\Omega(a)\right) \overset{?}{>} \log\left(\frac{a\cdot b}{(a+b)^2}\right);\ \Omega(b)\ge\frac{1}{2}\ln(b),\ \Omega(a)\ge\frac{1}{2}\ln(a)$$

$$a\cdot\Omega(b)+b\Omega(a)\ge\frac{1}{2}a\ln(b)+\frac{1}{2}b\ln(a)$$

Let us prove: $\frac{1}{\log(2)}\left(a\cdot\Omega(b)+b\cdot\Omega(a)\right) \overset{?}{>} \log\left(\frac{a\cdot b}{(a+b)^2}\right)$

$$\underbrace{\frac{1}{\log(2)}\left(a\cdot\Omega(b)+b\Omega(a)\right)}_{L_1}\ge\frac{1}{\log 2}\left(\frac{1}{2}a\cdot\ln b+\frac{1}{2}b\cdot\ln a\right)$$

$$L_1\ge\frac{1}{2}\cdot\frac{1}{\log 2}[a\cdot\ln b+b\cdot\ln a] \overset{?}{>} \log\left(\frac{a\cdot b}{(a+b)^2}\right)$$

$$\begin{matrix}0<b<1\\0<a<1\end{matrix}\Rightarrow b\cdot\log a>\log a\Rightarrow\frac{1}{2}b\cdot\log a>\frac{1}{2}\log a$$

Similarly: $\frac{1}{2}a\cdot\log b>\frac{1}{2}\log b$

$\frac{1}{2}(a\cdot\log b+b\cdot\log a)>\frac{1}{2}\log(a\cdot b)$. ***So:*** $\frac{1}{2\log 2}[a\cdot\log b+b\cdot\log a]>\frac{1}{2\log 2}\cdot\log(a\cdot b)$

Let us prove: $\frac{1}{2\cdot\log 2}\cdot\log(a\cdot b) \overset{?}{>} \log\left(\frac{a\cdot b}{(a+b)^2}\right)$

$$\frac{1}{2\log 2}\cdot\log(a\cdot b) \overset{?}{>} \log(a\cdot b)-\log(a+b)^2$$

$$\log(a+b)^2 \overset{?}{>} \log(a\cdot b)-\frac{1}{2\log 2}\cdot\log(a\cdot b)$$

$$\log(a+b)^2 \overset{?}{>} \log(a\cdot b)\left[1-\frac{1}{2\log 2}\right]$$

$$\frac{\log(a+b)^2}{\log(a\cdot b)} \overset{?}{<} 1-\frac{1}{2\log 2} \quad (*)$$

Because: $a\cdot b < (a+b)^2 \Rightarrow \log(a\cdot b) < \log(a+b)^2$

$$a, b \in]0,1[$$

$$1 > \frac{\log(a+b)^2}{\log(a\cdot b)} \Rightarrow \frac{\log(a+b)^2}{\log(a\cdot b)} - 1 < 0 < -\frac{1}{2\log 2}$$

Relation (*) is true. So:

$$\frac{1}{\log 2}\big(a\cdot\Omega(b)+b\Omega(a)\big) > \log\left(\frac{a\cdot b}{(a+b)^2}\right)$$

Solution 2 by Ali Jaffal-Lebanon

Let $g(x) = e^{-x}+x-1;\ D_g = [0,+\infty[$

$$g'(x) = -e^{-x}+1$$

x	0 $\qquad\qquad\qquad\qquad$ $+\infty$
$g'(x)$	+ + + + + + + + + + + + + + + + + +
$g(x)$	$0 \nearrow +\infty$

Then, $g(x) \geq 0$ ***for all*** $x \in [0,+\infty[$***. Therefore*** $e^{-x} \geq -x+1$ ***for all*** $x \in [0,+\infty[$

Let $f(x) = e^{-x}+x-\frac{x^2}{2}-1;\ D_f = [0,+\infty[$

We have $f'(x) = -e^{-x}-x+1 = -(e^{-x}+x-1) = -g(x)$***. Then:***

x	0 $\qquad\qquad\qquad\qquad$ $+\infty$
$f'(x)$	− − − − − − − − − − − − − − − − − −
$f(x)$	$0 \searrow -\infty$

Then, $f(x) \leq 0$ ***for all*** $x \geq 0$

Then we have: $-x+1 \leq e^{-x} \leq -x+1+\frac{x^2}{2}$ ***for all*** $x \in [0,+\infty[$ ***(*)***

Let $a \in]0,1[$ ***and*** $S_n = \sum_{k=1}^{k=n} a^{\frac{1}{k+n}} - n$. $S_n(a) = \sum_{k=1}^{k=n} e^{\frac{1}{k+n}\ln a} - n = \sum_{k=1}^{k=n} e^{-\frac{\ln\left(\frac{1}{a}\right)}{k+n}} - n$

We have: $\frac{+\ln\left(\frac{1}{a}\right)}{k+n} > 0$***, for*** $1 \leq k \leq n$ ***, since*** $\frac{1}{a} > 1$

Then by (*) we have: $1-\frac{\ln\left(\frac{1}{a}\right)}{k+n}\le e^{-\frac{\ln\left(\frac{1}{a}\right)}{k+n}}\le 1-\frac{\ln\left(\frac{1}{a}\right)}{k+n}+\frac{1}{2}\left(\frac{\ln\left(\frac{1}{a}\right)}{k+n}\right)^2$

So, $1+\frac{\ln a}{k+n}\le a^{\frac{1}{k+n}}\le 1+\frac{\ln a}{k+n}+\frac{1}{2}\left(\frac{\ln a}{k+n}\right)^2$

Then $\ln\sum_{k=1}^{k=n}\frac{1}{k+1}\le S_n(a)\le \ln\sum_{k=1}^{k=n}\frac{1}{k+n}+\frac{1}{2}(\ln a)^2\sum_{k=1}^{k=n}\left(\frac{1}{k+n}\right)^2$*. We have*

$$\lim_{n\to+\infty}\sum_{k=1}^{k=n}\frac{1}{k+n}=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{k=n}\frac{1}{1+\frac{k}{n}}=\int_0^1\frac{dx}{1+x}$$

$$\lim_{n\to+\infty}\frac{1}{n^2}\sum_{k=1}^{k=n}\frac{1}{\left(\frac{k}{n}+1\right)^2}=\lim_{n\to+\infty}\frac{1}{n}\times\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{k=n}\frac{1}{\left(\frac{k}{n}+1\right)^2}=0\times\int_0^1\frac{dx}{(x+1)^2}=0$$

Then $\lim_{n\to+\infty}S_a(a)=\ln a\times\ln 2$*. So,* $\Omega(a)=\ln a\times\ln 2$ *and* $\Omega(b)=\ln a\times\ln 2$

Let $\varphi(x)=x\ln x+\ln 2$ *when* $x\in s]0,+\infty[$*;* $\varphi'(x)=\ln x+1$

x	0 $\qquad\qquad$ e^{-1} $\qquad\qquad$ $+\infty$
$\varphi'(x)$	$-\ -\ -\ -\ -\ -\ -\ -\ -\ 0\ +\ +\ +\ +\ +\ +\ +\ +\ +$
$\varphi(x)$	$\ln 2\ \searrow\ -\frac{1}{e}+\ln 2\ \nearrow\ +\infty$

We have $-\frac{1}{e}+\ln 2>0$ *then* $\varphi(x)>0$*, for all* $x>0$*. Then* $x\ln x+\ln 2>0$

$x\ln x>-\ln 2$; $x^x>\frac{1}{2}$; $x^{-x}<2$; $x^{1-x}<2x$ *then* $\frac{a^{1-a}+b^{1-b}}{2}<a+b$

By (GM-AM) we have $\sqrt{a^{1-a}\cdot b^{1-b}}<\frac{a^{1-a}+b^{1-b}}{2}$

So, $\sqrt{a^{1-a}b^{1-b}}<a+b$; $a^{1-a}b^{1-b}<(a+b)^2$; $a^{a-1}b^{b-1}>\frac{1}{(a+b)^2}$

$$a^ab^b>\frac{ab}{(a+b)^2};\ a\ln a+b\ln b>\ln\left(\frac{ab}{(a+b)^2}\right)$$

$$\frac{a\Omega(a)}{\ln 2}+\frac{b\Omega(b)}{\ln 2}>\ln\left(\frac{ab}{(a+b)^2}\right);\ \frac{1}{\ln 2}\left(a\Omega(a)+b\Omega(b)\right)>\ln\left(\frac{ab}{(a+b)^2}\right)$$

783. If we define the function f for any positive integer n

$$f(n)=\int_0^{\infty}\tanh^{-1}(e^{-x})\,e^{-2nx}dx$$

then evaluate the sum

$$\sum_{n=1}^{\infty}\frac{f(n)}{n}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution 1 by Abdul Hafeez Ayinde-Nigeria

$$\text{Let } \Omega = \int_0^{\infty}\tanh^{-1}(e^{-x})\cdot e^{-2nx}dx$$

$$\Omega \overset{u=e^{-x}}{=} \int_0^1 \tanh^{-1}(u)\cdot u^{2n-1}\,du$$

$$\text{Let } \Lambda = \sum_{n=1}^{\infty}\frac{f(n)}{n}$$

$$\Lambda = \sum_{n=1}^{\infty}\frac{f(n)}{n} = \int_0^1 \frac{1}{u}\tanh^{-1}(u)\cdot\sum_{n=1}^{\infty}\frac{u^{2n}}{n}du$$

$$\Lambda = -\int_0^1 \frac{1}{u}\tanh^{-1}(u)\ln(1-u^2)\,du$$

$$\Lambda = -\int_0^1 \frac{\tanh^{-1}(u)\ln(1-u^2)}{u}du$$

$$\Lambda = -\frac{1}{2}\int_0^1\frac{\ln(1+u)\ln(1-u^2)}{u}du + \frac{1}{2}\int_0^1\frac{\ln(1-u)\ln(1-u^2)}{u}du$$

$$\Lambda = -\frac{1}{2}\int_0^1\frac{\ln^2(1+u)}{u}du - \frac{1}{2}\int_0^1\frac{\ln(1-u)\ln(1+u)}{u}du +$$

$$+\frac{1}{2}\int_0^1\frac{\ln(1-u)\ln(1+u)}{u}du + \frac{1}{2}\int_0^1\frac{\ln^2(1-u)}{u}du$$

$$\Lambda = -\frac{1}{2}\int_0^1\frac{\ln^2(1+u)}{u}du + \frac{1}{2}\int_0^1\frac{\ln^2(1-u)}{u}du$$

$$\Lambda = -\frac{1}{2}\int_1^2 \frac{\ln^2(u)}{u-1}du + \frac{1}{2}\int_0^1 \frac{\ln^2(u)}{1-u}du$$

$$\Lambda = \frac{1}{2}\int_1^2 \frac{\ln^2(u)}{1-u}du + \frac{1}{2}\int_0^1 \frac{\ln^2(u)}{1-u}du$$

$$\Lambda_1 = \frac{1}{2}\int_1^2 \frac{\ln^2(u)}{1-u}du$$

$$\Lambda_1 = \frac{1}{2}\int_1^{\frac{1}{2}} \frac{\ln^2(u)}{u(1-u)}du$$

$$\Lambda_1 = \frac{1}{2}\int_1^{\frac{1}{2}} \frac{\ln^2(u)}{u}du + \frac{1}{2}\int_1^{\frac{1}{2}} \frac{\ln^2(u)}{1-u}du$$

$$\Lambda_1 = \frac{1}{2}\int_1^{\frac{1}{2}} \ln^2(u)\,d(\ln u) + \frac{1}{2}\cdot\sum_{k=0}^{\infty}\int_1^{\frac{1}{2}} u^k \ln^2 u\,du$$

$$\Lambda_1 = \frac{1}{6}\ln^3(u)\Big|_1^{\frac{1}{2}} + \frac{1}{2}\cdot\frac{\partial^2}{\partial b^2}\Big|_{b=k}\sum_{k=0}^{\infty}\int_1^{\frac{1}{2}} u^b\,du$$

$$\Lambda_1 = \frac{-\ln^3(2)}{6} + \frac{1}{2}\cdot\frac{\partial^2}{\partial b^2}\Big|_{b=k}\sum_{k=0}^{\infty}\int_1^{\frac{1}{2}} u^b\,du$$

$$\Lambda_1 = \frac{-\ln^3(2)}{6} + \frac{1}{2}\cdot\frac{\partial^2}{\partial b^2}\Big|_{b=k}\sum_{k=0}^{\infty}\frac{u^{b+1}}{b+1}\Big|_1^{\frac{1}{2}}$$

$$\Lambda_1 = \frac{-\ln^3(2)}{6} + \frac{1}{2}\cdot\frac{\partial^2}{\partial b^2}\Big|_{b=k}\sum_{k=1}^{\infty}\left(\frac{\left(\frac{1}{2}\right)^b}{b} - \frac{1}{b}\right)$$

$$\Lambda_1 = \frac{-\ln^3(2)}{6} + \frac{1}{2}\cdot\frac{\partial}{\partial b}\Big|_{b=k}\sum_{k=1}^{\infty}\left(\frac{-\left(\frac{1}{2}\right)^b\ln(2)}{b} - \frac{\left(\frac{1}{2}\right)^b}{b^2} + \frac{1}{b^2}\right)$$

$$\Lambda_1 = \frac{-\ln^3(2)}{6} + \frac{1}{2}\Big|_{b=k}\sum_{k=1}^{\infty}\left(\frac{\left(\frac{1}{2}\right)^b \ln^2(2)}{b} + \frac{\left(\frac{1}{2}\right)^b \ln(2)}{b^2} + \frac{2\left(\frac{1}{2}\right)^b}{b^3} + \frac{\left(\frac{1}{2}\right)^b \ln(2)}{b^2} - \frac{2}{b^3}\right)$$

$$\Lambda_1 = \frac{-\ln^3(2)}{6} + \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{\left(\frac{1}{2}\right)^k \ln^2(2)}{k} + \frac{2\left(\frac{1}{2}\right)^k \ln(2)}{k^2} + \frac{2\left(\frac{1}{2}\right)^k}{k^3} - \frac{2}{k^3}\right)$$

$$\Lambda_1 = \frac{-\ln^3(2)}{6} + \frac{\ln^2(2)\, Li_1\left(\frac{1}{2}\right)}{2} + \ln(2)\, Li_2\left(\frac{1}{2}\right) + Li_3\left(\frac{1}{2}\right) - \zeta(3)$$

$$\Lambda_1 = \frac{-\ln^3(2)}{6} + \frac{\ln^3(2)}{2} + \ln(2)\left(\frac{\pi^2}{12} - \frac{\ln^2(2)}{2}\right) + Li_3\left(\frac{1}{2}\right) - \zeta(3)$$

$$\Lambda_1 = \frac{-\ln^3(2)}{6} + \frac{\pi^2 \ln 2}{12} + Li_3\left(\frac{1}{2}\right) - \zeta(3)$$

$$\Lambda_1 = \frac{-\ln^3(2)}{6} + \frac{\pi^2 \ln 2}{12} + \frac{7\zeta(3)}{8} - \frac{\pi^2 \ln 2}{12} + \frac{\ln^3(2)}{6} - \zeta(3) = \frac{-\zeta(3)}{8}$$

$$\Lambda_2 = \frac{1}{2}\int_0^1 \frac{\ln^2(u)}{1-u}\,du$$

$$\Lambda_2 = \frac{1}{2}\sum_{k=0}^{\infty}\int_0^1 u^k \ln^2(u)\,du$$

$$\Lambda_2 = \sum_{k=1}^{\infty}\frac{1}{k^3} = \zeta(3)$$

$$\Lambda = \Lambda_1 + \Lambda_2 = \zeta(3) - \frac{\zeta(3)}{8} = \frac{7\zeta(3)}{8}$$

Solution 2 by Kartick Chandra Betal-India

$$f(n) = \int_0^\infty \tanh^{-1}(e^{-x})\, e^{-2nx}dx, \textit{ let } e^{-x} = z, dx = -\frac{dz}{z}$$

$$= \int_1^0 \tanh^{-1}(z)\cdot z^{2n}\left(-\frac{dz}{z}\right) = \frac{1}{2}\int_0^1 z^{2n-1}\ln\left|\frac{1+z}{1-z}\right| dz$$

$$\sum_{n=1}^{\infty}\frac{f(n)}{n} = \frac{1}{2}\int_0^1 \ln\left(\frac{1+z}{1-z}\right)\cdot\sum_{n=1}^{\infty}\frac{(z^2)^n}{n}\cdot\frac{dz}{z} = -\frac{1}{2}\int_0^1 \ln\left(\frac{1+z}{1-z}\right)\ln(1-z^2)\frac{dz}{z}$$

$$= -\frac{1}{2}\int_0^1 \{\ln(1+z)-\ln(1-z)\}\{\ln(1+z)+\ln(1-z)\}\cdot\frac{dz}{z}$$

$$= -\frac{1}{2}\int_0^1 \frac{\ln^2(1+z)}{z}dz + \frac{1}{2}\int_0^1 \frac{\ln^2(1-z)}{z}dz = -\frac{1}{2}\int_1^2 \frac{\ln^2 z}{z-1}dz + \frac{1}{2}\int_0^1 \frac{\ln^2 z}{1-z}dz$$

$$= -\frac{1}{2}\int_{\frac{1}{2}}^1 \frac{\ln^2 z}{z(1-z)}dz + \frac{1}{2}\sum_{n=1}^{\infty}\int_0^1 z^{n-1}\ln^2 z\,dz$$

$$= -\frac{1}{2}\int_{\frac{1}{2}}^1 \frac{\ln^2 z}{z}dz - \frac{1}{2}\int_{\frac{1}{2}}^1 \frac{\ln^2 z}{1-z}dz + \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{z^n}{n}\ln^2 z - \frac{2z^n\ln z}{n^2} + \frac{2z^n}{n^3}\right]_0^1$$

$$= -\frac{1}{2}\left[\frac{\ln^3 z}{3}\right]_{\frac{1}{2}}^1 - \frac{1}{2}\left[\frac{z^n}{n}\ln^2 z - \frac{2z^n\ln z}{n^2} + \frac{2z^n}{n^3}\right]_{\frac{1}{2}}^1 + \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{z^n}{n}\ln^2 z - \frac{2z^n\ln z}{n^2} + \frac{2z^n}{n^3}\right]_0^1$$

$$= -\frac{1}{6}\ln^3 2 - \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{2}{n^3} - \left\{\frac{\ln^2 2}{n\cdot 2^n} + \frac{2\ln 2}{n^2\cdot 2^n} + \frac{2}{n^3\cdot 2^n}\right\}\right] + \zeta(3)$$

$$= -\frac{\ln^3 2}{6} - \frac{1}{2}\left[2\zeta(3) - \ln^3 2 - 2\ln 2\cdot Li_2\left(\frac{1}{2}\right) - 2Li_3\left(\frac{1}{2}\right)\right] + \zeta(3)$$

$$= -\frac{\ln^3 2}{6} - \zeta(3) + \frac{\ln^3 2}{2} + \ln 2\left\{\frac{\pi^2}{12} - \frac{\ln^2 2}{2}\right\} + \frac{\ln^3 2}{6} - \frac{\pi^2}{12}\ln 2 + \frac{7}{8}\zeta(3) + \zeta(3)$$

$$= +\frac{\ln^3 2}{2} + \frac{\pi^2}{12}\ln 2 - \frac{\ln^3 2}{2} - \frac{\pi^2}{12}\ln 2 + \frac{7}{8}\zeta(3) = \frac{7}{8}\zeta(3)$$

784. Inspired by Prof. Daniel Sitaru

Find:

$$\lim_{x\to\infty}\left\{\frac{\sqrt{4x^2+3}+\sqrt{x^2+x+1}+\sqrt{x^2-x+1}}{3\sqrt{(4x^2+3)(x^4+x^2+1)}}\right\}$$

where $\{\cdot\}$ denotes fractional part.

Proposed by Naren Bhandari-Bajura-Nepal

Solution by Adrian Popa-Romania

$$L=\lim_{x\to\infty}\left\{\frac{\sqrt{4x^2+3}+\sqrt{x^2+x+1}+\sqrt{x^2-x+1}}{3\sqrt{(4x^2+3)(x^4+x^2+1)}}\right\};\{x\}=x-[x]$$

$$\sqrt{4x^3+3}+\sqrt{x^2+x+1}+\sqrt{x^2-x+1}\overset{Dan\,Sitaru}{<}3\sqrt{(4x^2+3)(x^4+x+1)}\Rightarrow$$

$$\Rightarrow\frac{\sqrt{4x^2+3}+\sqrt{x^2+x+1}+\sqrt{x^2-x+1}}{3\sqrt{(4x^2+3)(x^4+x^2+1)}}<1\Rightarrow$$

$$\Rightarrow\left[\frac{\sqrt{4x^2+3}+\sqrt{x^2+x+1}+\sqrt{x^2-x+1}}{3\sqrt{(4x^3+3)(x^4+x^2+1)}}\right]=0\Rightarrow$$

$$\Rightarrow L=\lim_{x\to\infty}\frac{\sqrt{4x^2+3}+\sqrt{x^2+x+1}+\sqrt{x^2-x+1}}{3\sqrt{(4x^2+3)(x^4+x^2+1)}}=$$

$$=\lim_{x\to\infty}\frac{x\left(\sqrt{4+\frac{3}{x^2}}+\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}\right)}{3x^3\sqrt{\left(4+\frac{3}{x^2}\right)\left(1+\frac{1}{x^2}+\frac{1}{x^4}\right)}}=0$$

785. If $x,y,z>0$,

$$\Omega(x,y)=\lim_{n\to\infty}\left(\frac{\frac{n}{1^{\frac{x}{x+y}}}+\frac{n}{2^{\frac{x}{x+y}}}+\cdots+\frac{n}{n^{\frac{x}{x+y}}}}{1^{\frac{y}{x+y}}+2^{\frac{y}{x+y}}+\cdots+n^{\frac{y}{x+y}}}\right)$$

then:

$$\Omega(x,y)\cdot\Omega(y,z)\cdot\Omega(z,x)\geq 27$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Remus Florin Stanca – Romania

$$\Omega(x,y)\overset{Stolz\,Cesaro}{=}\lim_{n\to\infty}\frac{\frac{n+1}{1^{\frac{x}{x+y}}}+\frac{n+1}{2^{\frac{x}{3+y}}}+\cdots+\frac{n+1}{n^{\frac{x}{x+y}}}+\frac{n+1}{(n+1)^{\frac{x}{x+y}}}-\frac{n}{1^{\frac{x}{x+y}}}-\frac{n}{2^{\frac{x}{x+y}}}-\cdots-\frac{n}{n^{\frac{x}{x+y}}}}{1^{\frac{y}{x+y}}+2^{\frac{y}{x+y}}+\cdots+(n+1)^{\frac{y}{x+y}}-1^{\frac{y}{x+y}}-\cdots-n^{\frac{y}{x+y}}}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{1^{\frac{x}{x+y}}} + \frac{1}{2^{\frac{x}{x+y}}} + \cdots + \frac{1}{n^{\frac{x}{x+y}}} + \frac{n+1}{(n+1)^{\frac{x}{x+y}}}}{(n+1)^{\frac{y}{x+y}}} \overset{Stolz\ Cesaro}{=}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{(n+1)^{\frac{x}{x+y}}} + \frac{n+2}{(n+2)^{\frac{x}{x+y}}} - \frac{n+1}{(n+1)^{\frac{x}{x+y}}}}{(n+2)^{\frac{y}{x+y}} - (n+1)^{\frac{y}{x+y}}} =$$

$$= \lim_{n\to\infty} \frac{\frac{1}{(n+1)^{\frac{x}{x+y}}} + (n+2)^{\frac{y}{x+y}} - (n+1)^{\frac{y}{x+y}}}{(n+2)^{\frac{y}{x+y}} - (n+1)^{\frac{y}{x+y}}}$$

$$= 1 + \lim_{n\to\infty} \frac{\frac{1}{(n+1)^{\frac{x}{x+y}}}}{(n+1)^{\frac{y}{x+y}}\left(\left(\frac{n+2}{n+1}\right)^{\frac{y}{x+y}} - 1\right)} = 1 + \lim_{n\to\infty} \frac{1}{(n+1)\left(\left(\frac{n+2}{n+1}\right)^{\frac{y}{x+y}} - 1\right)} =$$

$$= 1 + \lim_{n\to\infty} \frac{\ln\left(\left(\frac{n+2}{n+1}\right)^{\frac{y}{x+y}}\right)}{\frac{y}{x+y}\ln\left(\frac{n+2}{n+1}\right)(n+1)\left(e^{\ln\left(\left(\frac{n+2}{n+1}\right)^{\frac{y}{x+y}}\right)} - 1\right)} =$$

$$= 1 + \frac{x+y}{y}\lim_{n\to\infty} \ln\left(\left(1 + \frac{1}{n+1}\right)^{n+1}\right) = 1 + \frac{x+y}{y} = 2 + \frac{x}{y} \Rightarrow$$

$$\Rightarrow \Omega(x,y) = 2 + \frac{x}{y}; \Omega(y,z) = 2 + \frac{y}{z}; \Omega(z,x) = 2 + \frac{z}{x}$$

$$\Rightarrow \Omega(x,y)\cdot\Omega(y,z)\cdot\Omega(z,x) = \left(2 + \frac{x}{y}\right)\left(2 + \frac{y}{z}\right)\left(2 + \frac{z}{x}\right)$$

$$= \frac{(2y+x)(2z+y)(2x+z)}{xyz} \geq$$

$$\overset{AM-GM}{\geq} \frac{3\sqrt[3]{xy^2}}{x}\cdot\frac{3\sqrt[3]{yz^2}}{y}\cdot\frac{3\sqrt[3]{zx^2}}{z} = \frac{27xyz}{xyz} = 27$$

Solution 2 by Nassim Nicholas Taleb-USA

$f = \dfrac{n\sum_{i=1}^{n}\frac{1}{i^{\frac{x}{x+y}}}}{\sum_{j=1}^{n} j^{\frac{y}{x+y}}}$. Let $a = \frac{x}{x+y}$, $f = \dfrac{Harmonic\ Mean(u^a)}{n\ Mean(u^a)}$, $0 < a < 1$

Using the integral approach for n large (since we are taking the limit),

which is simple by the result that $\int \frac{1}{u^a}du = \frac{u^{1-a}}{1-a}$ *convergent for* $a < 1$

$$\Omega(x,y) = \lim_{n\to\infty}\frac{n\int_1^n \frac{1}{u^a}du}{\int_1^n \frac{1}{u^{a-1}}du} = \lim_{n\to\infty}\frac{\frac{n^{1-a}(-n+n^a)}{-1+a}}{\frac{n^{-a}(-n^2+n^a)}{-2+a}} = \frac{a-2}{a-1} = \frac{x}{y}+2$$

$$g = \Omega(x,y)\cdot\Omega(y,z)\cdot\Omega(z,x) = \left(2+\frac{x}{y}\right)\left(2+\frac{y}{z}\right)\left(2+\frac{z}{x}\right) = \frac{(2y+x)(2z+y)(2x+z)}{xyz} \ge$$

$$\overset{AM-GM}{\ge} \frac{3\sqrt[3]{xy^2}}{x}\cdot\frac{3\sqrt[3]{yz^2}}{y}\cdot\frac{3\sqrt[3]{zx^2}}{z} = \frac{27xyz}{xyz} = 27$$

786. $x_1 = 3, n(x_1+x_2+\cdots+x_n) = x_n, n\in\mathbb{N}, n\ge 1$

Find:

$$\Omega = \sum_{n=1}^{\infty}(-1)^{n+1}x_n$$

Proposed by Daniel Sitaru – Romania

Solution by Ali Jaffal-Lebanon

We have: $n(x_1+x_2+\cdots+x_n) = x_n, n\ge 1$ *then* $x_1+x_2+\cdots+x_n = \frac{x_n}{n}$

$x_1+x_2+\cdots+x_{n-1} = \frac{x_{n-1}}{n-1}$ *for* $n\ge 2$ *then* $\frac{x_{n-1}}{n-1}+x_n = \frac{x_n}{n}$*. So,* $x_n = -\frac{n}{(n-1)^2}x_{n-1}$ *for*

$n\ge 2$*. So,* $x_{n+1} = -\frac{n+1}{n^2}x_n$ *for* $n\ge 1$*. We will prove that* $x_n = +\frac{3n(-1)^{n-1}\cdot n}{(n-1)!}$ *For* $n\ge 1$

By using mathematical induction: For $n = 1,2$ *we have* $x_1 = +3$ *and* $x_2 = -6$

So, it's true. Suppose that's true for $n = k\ge 1$*. So,* $x_k = \frac{3(-1)^{k-1}\cdot k}{(k-1)!}$

We will prove that is true for $n = k+1$

$x_{k+1} = -\frac{k+1}{k^2}, x_k = -\frac{k+1}{k^2}\cdot\frac{3(-1)^{k-1}}{(k-1)!}k$*. So,* $x_{k+1} = \frac{3(-1)^k(k+1)}{k!}$ *(true)*

Then, $x_n = \frac{3n(-1)^{n-1}}{(n-1)!}$ *for* $n\ge 1$*. Let* $S_n = \sum_{k=1}^{k=n}(-1)^{k+1}x_k = \sum_{k=1}^{k=n}\frac{3k(-1)^{k+1}\cdot(-1)^{k-1}}{(k-1)!}$

$$= \sum_{k=1}^{k=n}\frac{3k}{(k-1)!} = 3\sum_{k=0}^{k=n-1}\frac{k+1}{k!} = 3\sum_{k=0}^{k=n-1}\frac{k}{k!}+3\sum_{k=0}^{k=n-1}\frac{1}{k!}$$

$$\lim_{n\to+\infty} S_n = 3\sum_{0}^{\infty}\frac{k}{k!} + 3\sum_{k=0}^{\infty}\frac{1}{k!} = 3 + 3\sum_{k=1}^{\infty}\frac{k}{k!} + 3e$$

Since $\sum_{k=0}^{\infty}\frac{x^k}{k!} = e^x$ *for all* n.

$$= 3 + 3\sum_{k=1}^{\infty}\frac{1}{(k-1)!} + 3e = 3 + 3\sum_{k=1}^{\infty}\frac{1}{k!} + 3e = 3 + 6e$$

Solution 2 by Naren Bhandari-Bajura-Nepal

Here $x_1 = 3$ *and denote:*

$$\phi = \frac{x_n}{n} = \sum_{k=1}^{n} x_k \Rightarrow \phi - x_n = \sum_{k=1}^{n-1} x_k$$

and for $\frac{x_{n+1}}{n+1} - x_{n+1} = \sum_{k=1}^{n} x_k = \phi$. *Since* $\phi = \frac{x_n}{n}$ *thus we have:*

$$\frac{x_{n+1}}{n+1} - x_{n+1} = \frac{x_n}{n} \Rightarrow \frac{x_{n+1}}{x_n} = -\frac{n+1}{n^2}$$

Now we try to determine the x_n *in general. To do so we will be using the notation of telescoping product ie*

$$\prod_{m=1}^{n-1}\frac{x_{m+1}}{x_m} = \prod_{m=1}^{n-1}\left(-\left(\frac{m+1}{m^2}\right)\right); \forall n \ge 2$$

$\frac{x_n}{x_1} = \prod_{m=1}^{n-1}\frac{(-1)^{m+1}(m+1)}{m^2} \Rightarrow x_n = \frac{(-1)^{n+1}3n!}{\left((n-1)!\right)^2}$. *Thus*

$$\sum_{n=1}^{\infty}(-1)^{n+1}x_n = 3\sum_{n=1}^{\infty}\frac{n!}{\left((n-1)!\right)^2} = 3\sum_{n=1}^{\infty}\frac{n}{(n-1)!} = 3\sum_{n=1}^{\infty}\left(\frac{1}{(n-2)!} + \frac{1}{(n-1)!}\right) = 6e$$

Note: Final sum can be shown convergent by ratio test.

787. Find:

$$\Omega(a,b) = \lim_{n\to\infty}\left(n\left(\sum_{k=1}^{n}\left(a^{\frac{1}{n+k}}\cdot b^{\frac{2}{n+k}}\right) - n - log2\cdot log(ab^2)\right)\right), a,b > 1$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ali Jaffal-Lebanon

$$e^{\frac{1}{x}}=1+\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{6x^3}+\frac{1}{x^3}\varepsilon\left(\frac{1}{x}\right),\quad (*)\quad \lim_{x\to\infty}\varepsilon\left(\frac{1}{x}\right)=0$$

$\exists A,M>0$ ***such that for all*** $x>A$, $\left|\varepsilon\left(\frac{1}{x}\right)\right|<M$, $(**)$

Let $x=\left(\frac{logt}{n+k}\right)^{-1}$, $t=a^2b$. ***By (*):***

$$t^{\frac{1}{n+k}}=1+\frac{logt}{n+k}+\frac{1}{2}\left(\frac{logt}{n+k}\right)^2+\frac{1}{6}\left(\frac{logt}{n+k}\right)^3+\left(\frac{logt}{n+k}\right)^3\varepsilon\left(\frac{logt}{n+k}\right)$$

$$S_n(t)=n\sum_{k=1}^{n}t^{\frac{1}{n+k}}-n^2\cdot log2\cdot logt=$$

$$=nlogt\sum_{k=1}^{n}\frac{1}{n+k}-n\cdot log2\cdot logt+\frac{log^2t}{2}\sum_{k=1}^{n}\frac{n}{(n+k)^2}+$$

$$+\frac{log^3t}{6}\sum_{k=1}^{n}\frac{n}{(n+k)^2}+log^3t\sum_{k=1}^{n}\frac{n\varepsilon\left(\frac{logt}{n+k}\right)}{(n+k)^3}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{(n+k)^2}=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\left(1+\frac{k}{n}\right)^2}=\int_0^1\frac{dx}{(x+1)^2}=\frac{1}{2}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{(n+k)^3}=\lim_{n\to\infty}\left(\frac{1}{n}\cdot\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\left(1+\frac{k}{n}\right)^2}\right)=0\cdot\frac{1}{2}=0$$

By $(**)$ $\exists N\in\mathbb{N}^*$ ***such that:*** $\left|\varepsilon\left(\frac{logt}{n+k}\right)\right|\le M,\forall n\in\mathbb{N},n\ge N$ ***then:***

$$\left|\sum_{k=1}^{n}\frac{n\varepsilon\left(\frac{logt}{n+k}\right)}{(n+k)^3}\right|\le\sum_{k=1}^{N}\frac{n\left|\varepsilon\left(\frac{logt}{n+k}\right)\right|}{(n+k)^3}+\sum_{k=N+1}^{n}\frac{n\left|\varepsilon\left(\frac{logt}{n+k}\right)\right|}{(n+k)^3}\le$$

$$\le\sum_{k=1}^{N}\frac{n\left|\varepsilon\left(\frac{logt}{n+k}\right)\right|}{(n+k)^3}+M\sum_{k=1}^{n}\frac{n}{(n+k)^3}\Rightarrow\lim_{n\to\infty}\left|\sum_{k=1}^{n}\frac{n\varepsilon\left(\frac{logt}{n+k}\right)}{(n+k)^3}\right|=0$$

$$I_n=\sum_{k=1}^{n}\frac{1}{n+k}-log2,\ \lim_{n\to\infty}I_n=log2-log2=0$$

$$\lim_{n\to\infty} nI_n = \lim_{n\to\infty}\frac{I_{n+1}-I_n}{\frac{1}{n+1}-\frac{1}{n}} = \lim_{n\to\infty}\frac{-\frac{1}{n+1}+\frac{1}{2n+1}+\frac{1}{2n+2}}{-\frac{1}{n(n+1)}} = -\frac{1}{4}$$

$$\lim_{n\to\infty} S_n(t) = -\frac{1}{4}logt+\frac{1}{4}log^2t$$

$$\Omega(a,b) = -\frac{1}{4}log(ab^2)+\frac{1}{4}log^2(ab^2)$$

Solution 2 by Florentin Vișescu-Romania

$$\Omega(a,b) = \lim_{n\to\infty}\left(n\left(\sum_{k=1}^{n}\left(a^{\frac{1}{n+k}}\cdot b^{\frac{2}{n+k}}\right)-n-log2\cdot log(ab^2)\right)\right) =$$

$$= \lim_{n\to\infty}\left(n\left(\sum_{k=1}^{n}(ab^2)^{\frac{1}{n+k}}-n-log2\cdot log(ab^2)\right)\right)\ (Not\ c = ab^2) =$$

$$= \lim_{n\to\infty}\left(n\left(\sum_{k=1}^{n}(c)^{\frac{1}{n+k}}-n-log2\cdot log(c)\right)\right) = \lim_{n\to\infty}\frac{\sum_{k=1}^{n}(c)^{\frac{1}{n+k}}-n-log2\cdot log(c)}{\frac{1}{n}} =$$

$$\overset{TCS}{=} \lim_{n\to\infty}\frac{c^{\frac{1}{2n+1}}+c^{\frac{1}{2n+2}}-1-c^{\frac{1}{n+1}}}{\frac{-1}{n^2+n}} = \lim_{n\to\infty}\frac{c^{\frac{1}{2n+1}}+c^{\frac{1}{2n+2}}-1-c^{\frac{1}{n+1}}}{\frac{-1}{n^2+n}\cdot\frac{n^2}{n^2}} =$$

$$= \lim_{n\to\infty}\frac{-c^{\frac{1}{2n+1}}-c^{\frac{1}{2n+2}}+1+c^{\frac{1}{n+1}}}{\frac{1}{n^2}} = \lim_{\substack{x\to 0\\ x>0}}\frac{c^{\frac{x}{x+1}}+1-c^{\frac{x}{x+2}}-c^{\frac{x}{2x+2}}}{x^2} =$$

$$\overset{L'H}{=} \lim_{\substack{x\to 0\\ x>0}}\frac{\frac{1}{(x+1)^2}e^{\frac{x}{x+1}}c^{\frac{x}{x+1}}logc+1-\frac{2}{(x+2)^2}e^{\frac{x}{x+2}}c^{\frac{x}{x+2}}logc-\frac{2}{(2x+2)^2}e^{\frac{x}{2x+2}}c^{\frac{x}{2x+2}}logc}{2x}$$

$$\overset{L'H}{=} \frac{logc\left(logc-2-logc-\frac{4}{16}logc+\frac{4}{8}-logc\cdot\frac{4}{24}+\frac{8}{8}\right)}{2} =$$

$$= logc\left(\frac{1}{2}logc-\frac{1}{2}\right)^2 = \frac{logc(logc-1)}{4} = \frac{log(ab^2)(log(ab^2)-1)}{4}$$

788. $\gamma_n = H_n - \log n,\ n\geq 1,\ x\in\mathbb{R}$. **Find:**

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$$\Omega(x) = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\gamma_i^{\sin^2 x}\cdot\gamma_{n+1-i}^{\cos^2 x}\right)\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ali Jaffal-Lebanon

Let $(a_n)_n : (b_n)_n$ *be two sequence so that* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to+\infty} b_n = b$

We will prove that $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{k=n} a_k\, b_{n+1-k} = ab$. *We have*

$$\frac{1}{n}\sum_{k=1}^{k=n}(a_k b_{n+1-k} - ab) = \frac{1}{n}\sum_{k=1}^{k=n}(a_k - a)\, b_{n+1-k} + \frac{a}{n}\sum_{k=1}^{k=n}(b_n - b) + \frac{b_{n+1}\cdot a}{n}$$

$\exists M > 0$ *so that* $|b_n| \le M, \forall n \in \mathbb{N}^*$. *By Cesaro theorem:*

$$0 \le \frac{1}{n}\left|\sum_{k=1}^{k=n}(a_k - a)b_{n+1-k}\right| \le \frac{M}{n}\sum_{k=1}^{k=1}|a_k - a|$$

We have $\lim_{n\to\infty}\frac{M}{n}\sum_{k=1}^{k=n}|a_k - a| = 0$. *So,* $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{k=n}(a_k - a)\, b_{n+1-k} = 0$

We have also that: $\lim_{n\to\infty}\frac{a}{n}\sum_{k=1}^{k=n}(b_n - b) = 0$ *then:*

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{k=n} a_k\, b_{n+1-k} = ab \quad (**)$$

Let $a_n = \gamma_n^{\sin^2 x}$ *and* $b_n = \gamma_n^{\cos^2 x}$

We have $\lim_{n\to+\infty} a_n = \gamma^{\sin^2 x}$ *and* $\lim_{n+\infty} b_n = \gamma^{\cos^2 x}$

So, by (**): $\Omega(x) = \lim_{n\to+\infty}\frac{1}{n}\sum_{i=1}^{i=n} a_i\, b_{n+1-i} = \gamma^{\sin^2 x + cos^2 x} = \gamma$

Solution 2 by Naren Bhandari-Bajura-Nepal

For convenience, we write $\sin^2 x = S(x)$ *and* $\cos^2 x = C(x)$ *then we observe that:*

$$\psi(n) = \sum_{i=1}^{n}\gamma_i^{S(x)}\cdot\gamma_{n+1-i}^{C(x)} = \sum_{i=1}^{n}\sum_{k=n+1-i}^{n}\gamma_i^{S(x)}\gamma_k^{C(x)} = \sum_{i=1}^{n}\gamma_i^{S(x)}\sum_{k=1}^{n}\gamma_k^{C(x)}$$

$= \sum_{m=1}^{n}\gamma_m^{S(x)+C(x)} = \sum_{m=1}^{n}\gamma_m$ *(for* $i = k = m$*). Since* $S(x) + C(x) = 1$. *Now,*

$$\lim_{n\to\infty}\frac{\psi(n)}{n} = \lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}\gamma_m = \lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}(H_m - \log m)$$

$$= \lim_{n\to\infty} \frac{1}{n}\left(\sum_{m=1}^{n}\sum_{j=1}^{n}\frac{1}{j} - \log\left(\prod_{m=1}^{n} m\right)\right) = \lim_{n\to\infty} \frac{1}{n}\left(\sum_{m=1}^{n}\frac{n-m+1}{m} - \log(n!)\right)$$

$$= \lim_{n\to\infty}\left(\sum_{m=1}^{n}\frac{1}{m} - \frac{n\log n - n + O\log(n)}{n} - 1 + \frac{1}{n}\sum_{m=1}^{n}\frac{1}{m}\right)$$

$$= \lim_{n\to\infty}\left(\sum_{m=1}^{n}\frac{1}{m} - \log n\right) + 0 = \gamma$$

789.

$$\Delta_n = \begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2^3 & 3^3 & \cdots & n^3 \\ 1 & 2^5 & 3^5 & \cdots & n^5 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2^{2n-1} & 3^{2n-1} & \cdots & n^{2n-1} \end{vmatrix}, n \geq 2$$

Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{\Delta_{n+1}}{n^{2n+1}\cdot\Delta_n}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu – Romania

First, we will prove $\Delta_n = 1!\cdot 3!\cdot\ldots\cdot(2n-1)!\ (1)$ ***by mathematical induction.***

For $P(2): \Delta_2 = \begin{vmatrix} 1 & 2 \\ 1 & 2^3 \end{vmatrix} = 6$ ***true. Let*** $P(n)$ ***true and we will prove*** $P(n+1)$

$$\text{Let } f(x) = \begin{vmatrix} 1 & 2 & \cdots & n & x \\ 1^3 & 2^3 & \cdots & n^3 & x^3 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1^{2n+1} & 2^{2n+1} & \cdots & n^{2n+1} & x^{2n+1} \end{vmatrix}$$

$f \in \mathbb{R}[x]$ ***and grad*** $f = 2n+1$ ***and have*** $0, \pm 1, \pm 2, \ldots, \pm n$ ***roots*** $\Rightarrow$

$$f(x) = a_{2n+1}(x-x_1)(x-x_2)\ldots(x-x_{2n+1}) \Rightarrow$$

$$f(x) = 1!\,3!\cdots(2n-1)!\,x(x^2-1)(x^2-2^2)\cdots(x^2-n^2) \Rightarrow$$

$\Delta_{n+1} = 1!\,3!\cdots(2n-1)!\,(2n+1)! \Rightarrow P(n+1)$ ***true.***

From (1)$\Rightarrow \Omega = \lim_{n\to\infty}\frac{\Delta_{n+1}}{n^{2n+1}\Delta_n} = \lim_{n\to\infty}\frac{(2n+1)!}{n^{2n+1}}$ ***(2)***

Let $x_n = \frac{(2n+1)!}{n^{2n+1}}$

$$\lim_{n\to\infty}\frac{x_{n+1}}{x_n} = \lim_{n\to\infty}\frac{(2n+3)!}{(n+1)^{2n+3}}\cdot\frac{n^{2n+1}}{(2n+1)!} = \lim_{n\to\infty}\frac{(2n+3)(2n+2)}{(n+1)^2}\cdot\frac{n^{2n+1}}{(n+1)^{2n+1}} =$$

$$= \lim_{n\to\infty} 4\cdot\left(\frac{n}{n+1}\right)^{2n+1} = 4\lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^n\right]^{\frac{2n+1}{4}} = \frac{4}{e^2} < 1 \quad \textbf{\textit{(3)}}$$

From (2)+(3)$\Rightarrow \Omega = 0.$

Solution 2 by Remus Florin Stanca – Romania

$$\Delta_n = n!\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2^2 & 3^2 & \cdots & n^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2^{2n-2} & 3^{2n-2} & \cdots & n^{2n-2} \end{vmatrix}$$

$$= n!\cdot\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 2^2-1 & 3^2-1 & \cdots & n^2-1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2^{2n-2}-1 & 3^{2n-2}-1 & \cdots & n^{2n-2}-1 \end{vmatrix}$$

$$= n!\begin{vmatrix} 2^2-1 & 3^2-1 & \cdots & n^2-1 \\ 2^4-1 & 3^4-1 & \cdots & n^4-1 \\ \cdots & \cdots & \cdots & \cdots \\ 2^{2n-2}-1 & 3^{2n-2} & \cdots & n^{2n-2}-1 \end{vmatrix} =$$

$$= n!\,(2^2-1)(3^2-1)\cdot\ldots\cdot(n^2-1)\cdot$$

$$\cdot\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 2^2+1 & 3^2+1 & \cdots & n^2+1 \\ \cdots & \cdots & \cdots & \cdots \\ 2^{2n-4}+..+1 & 3^{2n-4}+\cdots+1 & \cdots & n^{2n-4}+\cdots+1 \end{vmatrix} =$$

$$= n!\cdot(2^2-1)(3^2-1)\cdot\ldots\cdot(n^2-1)\cdot\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 2^2 & 3^2 & \cdots & n^2 \\ \cdots & \cdots & \cdots & \cdots \\ 2^{2n-4} & 3^{2n-4} & \cdots & n^{2n-4} \end{vmatrix}$$

$$= n!\cdot 2^2\cdot\ldots\cdot 2^{2n-4}\cdot(2^2-1)(3^2-1)\cdot\ldots\cdot(n^2-1)\cdot\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & \left(\frac{3}{2}\right)^2 & \cdots & \left(\frac{n}{2}\right)^2 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \left(\frac{3}{2}\right)^{2n-4} & \cdots & \left(\frac{n}{2}\right)^{2n-4} \end{vmatrix}$$

$$= n!\cdot 2^2\cdot\ldots\cdot 2^{2n-4}(2^2-1)(3^2-1)\cdot\ldots\cdot(n^2-1)\cdot\begin{vmatrix}1 & 0 & \cdots & 0\\ 1 & \left(\frac{3}{2}\right)^2-1 & \cdots & \left(\frac{n}{2}\right)^2-1\\ \ldots & \ldots & \ldots & \ldots\\ 1 & \left(\frac{3}{2}\right)^{2n-4}-1 & \cdots & \left(\frac{n}{2}\right)^{2n-4}-1\end{vmatrix}$$

$$= n!\cdot 2^2\cdot\ldots\cdot 2^{2n-4}(2^2-1)(3^2-1)\cdot\ldots\cdot(n^2-1)\cdot\begin{vmatrix}\left(\frac{3}{2}\right)^2-1 & \cdots & \left(\frac{n}{2}\right)^2-1\\ \ldots & \ldots & \ldots\\ \left(\frac{3}{2}\right)^{2n-4}-1 & \cdots & \left(\frac{n}{2}\right)^{2n-4}-1\end{vmatrix}$$

$$= n!\cdot 2^2\cdot\ldots\cdot 2^{2n-4}\cdot(2^2-1)(3^2-1)\cdot\ldots\cdot(n^2-1)\cdot\left(\left(\frac{3}{2}\right)^2-1\right)\cdot\ldots\cdot\left(\left(\frac{n}{2}\right)^2-1\right)\cdot$$

$$\cdot\begin{vmatrix}1 & \cdots & 1\\ \ldots & \ldots & \ldots\\ \left(\frac{3}{2}\right)^{2n-6} & \cdots & \left(\frac{n}{2}\right)^{2n-6}\end{vmatrix} = n!\cdot 2^2\cdot\ldots\cdot 2^{2n-4}\cdot(2^2-1)(3^2-1)\cdot\ldots\cdot(n^2-1)\cdot$$

$$\cdot\left(\left(\frac{3}{2}\right)^2-1\right)\cdot\ldots\cdot\left(\left(\frac{n}{2}\right)^2-1\right)\cdot\left(\frac{3}{2}\right)^2\cdot\ldots\cdot\left(\frac{3}{2}\right)^{2n-6}\cdot\begin{vmatrix}1 & 1 & \cdots & 1\\ 1 & \left(\frac{n}{3}\right)^2 & \cdots & \left(\frac{n}{3}\right)^2\\ \ldots & \ldots & \ldots & \ldots\\ 1 & \left(\frac{4}{3}\right)^{2n-6} & \cdots & \left(\frac{n}{2}\right)^{2n-6}\end{vmatrix}$$

$$= \cdots = n!\left(\left(\left(\frac{2}{1}\right)^2-1\right)\left(\left(\frac{3}{1}\right)^2-1\right)\cdot\ldots\cdot\left(\left(\frac{n}{1}\right)^2-1\right)\cdot\ldots\cdot\left(\left(\frac{n-1}{n-2}\right)^2-1\right)\cdot\left(\left(\frac{n}{n-2}\right)^2-1\right)\right)\cdot$$

$$\cdot\left(\left(\frac{2}{1}\right)^2\cdot\left(\frac{2}{1}\right)^4\cdot\ldots\cdot\left(\frac{2}{1}\right)^{2n-4}\left(\frac{3}{2}\right)^2\cdot\ldots\cdot\left(\frac{3}{2}\right)^{2n-6}\cdot\ldots\cdot\left(\frac{n-2}{n-3}\right)^2\cdot\left(\frac{n-2}{n-3}\right)^4\right)\cdot$$

$$\cdot\begin{vmatrix}1 & 1\\ \left(\frac{n-1}{n-2}\right)^2 & \left(\frac{n}{n-2}\right)^2\end{vmatrix} = n!\cdot\frac{2^{S_{2n-4}}\cdot 3^{S_{2n-6}}\cdot\ldots\cdot(n-2)^{S_4}}{2^{S_{2n-6}}\cdot 3^{S_{n-8}}\cdot\ldots\cdot(n-3)^{S_4}}\cdot\frac{2n-1}{(n-2)^2}\cdot$$

$$\cdot\frac{(n-1)!\cdot(n-2)!\cdot\ldots\cdot 2!(n+1)!(n+2)!\cdot\ldots\cdot(2n-2)!}{2!\cdot 4!\cdot\ldots\cdot(2n-4)!}\cdot\frac{1}{1^{2n-2}\cdot 2^{2n-4}\cdot\ldots\cdot(n-2)^4}$$ ***and*** $S_{2n} = 2+4+\cdots+2n$

$$= n!\cdot 2^{2n-4}\cdot 3^{2n-6}\cdot\ldots\cdot(n-3)^6(n-2)^6\cdot\frac{(2n-1)}{(n-2)^2}\cdot$$

$$\cdot\frac{(n-1)!\,(n-2)!\cdot\ldots\cdot 2!\,(n+1)!\,(n+2)!\cdot\ldots\cdot(2n-2)!}{2!\cdot 4!\cdot\ldots\cdot(2n-4)!}\cdot\frac{1}{2^{2n-4}\cdot\ldots\cdot(n-2)^4} =$$

$$= n!\cdot(n-2)^2\cdot\frac{(2n-1)}{(n-2)^2}\cdot(n-1)!\cdot(n-2)!\cdot\ldots\cdot 2!\cdot(n+1)!\,(n+2)!\cdot\ldots\cdot(2n-2)!\cdot\frac{1}{2!\cdot 4!\cdot\ldots\cdot(2n-4)!}$$

$$= \frac{2!\cdot 3!\cdot \ldots \cdot (2n-2)!\cdot (2n-1)}{2!\cdot 4!\cdot \ldots \cdot (2n-4)} = 1!\cdot 3!\cdot 5!\cdot \ldots \cdot (2n-1)! = \Delta_n$$

$$\Rightarrow \Omega = \lim_{n\to\infty} \frac{1!\cdot 3!\cdot \ldots \cdot (2n+1)!}{n^{2n+1}\cdot 1\cdot 3!\cdot \ldots \cdot (2n-1)!} = \lim_{n\to\infty} \frac{(2n+1)!}{n^{2n+1}} = ?$$

$$\textit{Let } x_n = \frac{(2n+1)!}{n^{2n+1}} \Rightarrow \frac{x_{n+1}}{x_n} = \frac{(2n+3)!}{(n+1)^{2n+3}}\cdot \frac{n^{2n+1}}{(2n+1)!} \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} (2n+2)(2n+3)\cdot \frac{1}{e^2}\cdot \frac{1}{(n+1)^2} =$$

$$= \lim_{n\to\infty} \frac{n}{e^2} < 1 \Rightarrow \lim_{n\to\infty} x_n = 0 \Rightarrow \Omega = 0$$

790. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{n^3}\sum_{1\le i<j\le n}\left((-1)^{i+j}\cdot i\cdot j\right)\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Adrian Popa-Romania

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{n^3}\sum_{1\le i<j\le n}\left((-1)^{i+j}\cdot i\cdot j\right)\right) = \frac{1}{2}\lim_{n\to\infty}\left[\frac{1}{n^3}\left(-1+2-3+4-5+6-\cdots+(-1)^n n\right)^2 - \sum_{k=1}^{n} k^2\right]$$

$\therefore$***We've used the following relationship:*** $(a+b)^2 = a^2 + 2ab + b^2 \Rightarrow$

$$\Rightarrow 2ab = (a+b)^2 - a^2 - b^2 \Rightarrow ab = \frac{(a+b)^2 - a^2 - b^2}{2} \therefore$$

$$2\Omega = \underbrace{\lim_{n\to\infty}\frac{\left(\overbrace{-1+2}^{1} - \overbrace{3+4}^{1} - \cdots + (-1)^n n\right)^2}{n^3}}_{L_1} - \underbrace{\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k^2}{n^2}}_{L_2}$$

We evaluate L_1 ***using sandwich theorem:*** $\underset{\substack{\downarrow\\ 0}}{\frac{\overbrace{1+1+\cdots+1}^{\left[\frac{n}{2}\right]\,times}-n}{n^3}} < L_1 < \underset{\substack{\downarrow\\ 0}}{\frac{\frac{n}{2}\,times}{n^3}} \Rightarrow L_1 \to 0$

We calculate L_2: $L_2 = \int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$

$$2\Omega = 0 - \frac{1}{3} = -\frac{1}{3} \Rightarrow \Omega = -\frac{1}{6}$$

Solution 2 by Naren Bhandari-Bajura-Nepal

Denote:

$$L = \lim_{n\to\infty} \frac{1}{n^3} \sum_{1\le i<j\le n} (-1)^{i+j}\, ij = \lim_{n\to\infty} \sum_{j=1}^{n} \sum_{i\ne 1}^{n} (-1)^{i+j} ij$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \sum_{j=1}^{n} \sum_{i=j+1}^{n} (-1)^{i+j} = \lim_{n\to\infty} \frac{1}{n^3} \left(\sum_{j=1}^{n} (-1)^j j\right)\left(\sum_{i=1}^{n} (-1)^i j\right)$$

$$-\lim_{n\to\infty} \frac{1}{n^3} \left(\sum_{j=1}^{n} \sum_{i=1}^{n} (-1)^j j((-1)^i i)\right)$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \left(\sum_{k=1}^{n} (2k+1) - (2k)\right)^2 - \lim_{n\to\infty} \frac{1}{n^3} \sum_{j=1}^{n} \sum_{i=j+1}^{n} (-1)^{j+1} ij$$

$$-\lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2 \; (i = j = k)$$

$$= \lim_{n\to\infty} \frac{n^2}{n^3} - \lim_{n\to\infty} \frac{1}{n^3} \sum_{1\le i<j\le n} (-1)^{i+j}\, ij - \lim_{n\to\infty} \frac{1}{n^3} \left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$2 \lim_{n\to\infty} \frac{1}{n^3} \sum_{1\le i<i\le n} (-1)^{i+j}\, ij = -\frac{2}{6} \Rightarrow L = -\frac{1}{6}$$

Solution 3 by Remus Florin Stanca-Romania

Let $n = 2k+1 \Rightarrow \Omega \overset{\text{Stolz Cesaro}}{=} \lim_{n\to\infty} \frac{\begin{matrix} 1(-2+3-\dots+n-(n+1))+2(-3+4-\dots-n+(n+1))+\dots+ \\ +(n-2)(-(n-1)+n+1+1)+(n-1)(-n+n+1)+n(-(n+1))- \\ -1(-2+\dots+n)-2(-3+\dots-n)-\dots-(n-1)(-(n)) \end{matrix}}{3n^2+3n+1} =$

$$= \lim_{n\to\infty} \frac{(n+1)(-1+2-\dots-n)}{3n^2+3n+1} =$$

$$= \lim_{n\to\infty} \frac{(n+1)\left(2\left(1+\dots+\frac{n-1}{2}\right) - (1+3+\dots+n)\right)}{3n^2+3n+1} =$$

$$= \lim_{n\to\infty} \frac{(n+1)\left(\frac{n-1}{2}\left(\frac{n-1}{2}+1\right)-\left(\frac{n+1}{2}\right)^2\right)}{3n^2+3n+1} =$$

$$= \lim_{n\to\infty} \frac{(n+1)\left(\left(\frac{n-1}{2}\right)^2+\frac{n-1}{2}-\left(\frac{n+1}{2}\right)^2\right)}{3n^2+3n+1} = \lim_{n\to\infty} \frac{(n+1)\left(\frac{n-1}{2}-n\right)}{3n^2+3n+1} =$$

$= \lim_{n\to\infty} -\frac{(n+1)^2}{6n^2+6n+2} = -\frac{1}{6}$ ***and if*** $n = 2k$ ***we have the same result*** $\Rightarrow \Omega = -\frac{1}{6}$

Solution 4 by Florentin Vișescu-Romania

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{1\le i<j\le n} (-1)^i i(-1)^j j = \lim_{n\to\infty} \frac{\sum_{1\le i<j\le n}(-1)^i i(-1)^j j}{n^3} \overset{CS}{=}$$

$$\lim_{n\to\infty} \frac{\sum_{1\le i<j\le n}(-1)^i i\,(-1)^j j - \sum_{1\le i<j\le n}(-1)^i i(-1)^j j}{(n+1)^3-n^3} =$$

$$= \lim_{n\to\infty} \frac{\sum_{i=1}^{n}(-1)^i i\,(-1)^{n+1}(n+1)}{3n^2+3n+1} = \lim_{n\to\infty} \underbrace{\frac{(-1)^{n+1}(n+1)\sum_{i=1}^{n}(-1)^i j}{3n^2+3n+1}}_{x_n}$$

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} \frac{(-1)^{2n+1}(2n+1)\sum_{i=1}^{2n}(-1)^i i}{12n^2+6n+1} =$$

$$= \lim_{n\to\infty} \frac{-(2n+1)\left(-\overbrace{1+2}^{1}-\overbrace{3+4}^{1}+\cdots+\overbrace{2n}^{1}\right)}{12n^2+6n+1} = \lim_{n\to\infty} \frac{-(2n+1)n}{12n^2+6n+1} = -\frac{1}{6}$$

$$\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} \frac{(-1)^{2n+2}(2n+2)\sum_{i=1}^{2n+1}(-1)^i i}{3(2n+1)^2+3(2n+1)+1}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(-1+2-3+4+\cdots+2n-(2n+1))}{3(2n+1)^2+3(2n+1)+1} = \lim_{n\to\infty} \frac{(2n+2)(-n-1)}{3(2n+1)^2+3(2n+1)+1} = -\frac{1}{6}$$

$$\Omega = -\frac{1}{6}$$

791. Find:

$$\Omega = \lim_{n\to\infty} \left(\sum_{k=1}^{n}\left(\sum_{i=1}^{k} i(i+1)^2(i+2)\right)^{-1}\right)$$

Proposed by Vasile Mircea Popa – Romania

Solution 1 by Mokhtar Khassani-Mostaganem-Algerie

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\sum_{i=1}^{k}i(i+1)^2(i+2)\right)^{-1}=\sum_{k=1}^{\infty}\left(\sum_{i=1}^{k}(i^4+4i^3+5i^2+2i)\right)=$$

$$=\sum_{k=1}^{\infty}\left(\frac{k(k+1)(2k+1)(3k^2+3k-1)}{30}+4\frac{k^2(k+1)^2}{4}+\atop +5\frac{k(k+1)(2k+1)}{6}+2\frac{k(k+1)}{2}\right)^{-1}$$

$$=10\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)(k+3)(2k+3)}=$$

$$=5\sum_{k=1}^{\infty}\left(\frac{1}{18k}+\frac{8}{9\left(k+\frac{3}{2}\right)}+\frac{1}{18(k+3)}-\frac{1}{2(k+1)}-\frac{1}{2(k+2)}\right)$$

$$=10\sum_{k=1}^{\infty}\left(\frac{8}{9}\left(\frac{1}{k+\frac{3}{2}}-\frac{1}{k}\right)+\frac{1}{18}\left(\frac{1}{k+3}-\frac{1}{k}\right)+\frac{1}{2}\left(\frac{1}{k}-\frac{1}{k+1}\right)+\frac{1}{2}\left(\frac{1}{k}-\frac{1}{k+2}\right)\right)=$$

$$=10\left(\frac{8}{9}\left(-\gamma-\Psi\left(\frac{5}{2}\right)\right)+\frac{1}{18}\left(-\frac{11}{6}\right)+\frac{1}{2}+\frac{3}{4}\right)=\frac{10}{9}(16\log 2-11)$$

Note: $\Psi(1+z)=-\gamma+\sum_{n=1}^{\infty}\frac{z}{n(n+z)},\ z\neq-1,-2,-3,\dots,\ \Psi\left(\frac{5}{2}\right)=2\left(\frac{4}{3}-\log 2\right)-\gamma,$

Ψ: ***the digamma function***

Solution 2 by Nelson Javier Villaherrera Lopez-El Salvador

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\sum_{i=1}^{k}i(i+1)^2(i+2)\right]^{-1}=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\sum_{i=1}^{k}i(i^2+2i+1)(i+2)\right]^{-1}=$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\sum_{i=1}^{k}i(i^3+4i^2+5i+2)\right]^{-1}=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\sum_{i=1}^{k}i^4+4\sum_{i=1}^{k}i^3+5\sum_{i=1}^{k}i^2+2\sum_{i=1}^{k}i\right)^{-1}$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{k(k+1)(2k+1)(3k^2+3k-1)}{30}+4\frac{k^2(k+1)^2}{4}+\atop +5\frac{k(k+1)(2k+1)}{6}+2\frac{k(k+1)}{2}\right]^{-1}$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{k(k+1)(2k+1)(3k^2+3k-1)+30k^2(k+1)^2+25k(k+1)(2k+1)+30k(k+1)}{30}\right]^{-1}$$

$$= 30\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)[(2k+1)(3k^2+3k-1)+30k(k+1)+25(2k+1)+30]} =$$

$$= 30\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)[(2k+1)(3k^2+3k+24)+30(k^2+k+1)]}$$

$$= 10\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)[(2k+1)(k^2+k+8)+10(k^2+k+1)]} =$$

$$= 10\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)(2k^3+3k^2+17k+8+10k^2+10k+10)}$$

$$= 10\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)(2k^3+13k^2+27k+18)} =$$

$$= 10\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)(k^3+6k^2+12k+8+k^3+7k^2+15k+10)}$$

$$= 10\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)[(k+2)^3+k^3+6k^2+12k+8+k^2+3k+2]} =$$

$$= 10\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)[(k+2)^3+(k+2)^3+(k+2)(k+1)]}$$

$$= 10\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)[2(k+2)^2+k+1]} =$$

$$= 10\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)[2(k+2)^2+(k+2)-1]} =$$

$$= 20\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)[2^2(k+2)^2+2(k+2)-2]}$$

$$= 20\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)[2(k+2)+2][2(k+2)-1]} =$$

$$= 10\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)(k+3)(2k+3)} =$$

$$= \frac{5}{9}\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{9}{k+1}-\frac{9}{k+2}+\frac{1}{k+3}+\frac{32}{2k+3}\right)$$

$$= \frac{5}{9}\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{9}{k+1}-\frac{9}{k+2}+\frac{1}{k+3}+\frac{32}{2k+3}\right) =$$

$$= \frac{5}{9}[16\ln(4)-22] = \frac{10}{9}[16\ln(2)-11]$$

$$\frac{1}{k(k+1)(k+2)(k+3)(2k+3)} = \frac{A}{k}+\frac{B}{k+1}+\frac{C}{k+2}+\frac{D}{k+3}+\frac{E}{2k+3} =$$

$$= \frac{\frac{1}{18}}{k}-\frac{\frac{1}{2}}{k+1}-\frac{\frac{1}{2}}{k+2}+\frac{\frac{1}{18}}{k+3}+\frac{\frac{16}{9}}{2k+3} = \frac{1}{18}\left(\frac{1}{k}-\frac{9}{k+1}-\frac{9}{k+2}+\frac{1}{k+3}+\frac{32}{2k+3}\right)$$

$$1 = A(k+1)(k+2)(k+3)(2k+3)+Bk(k+2)(k+3)(2k+3)+$$

$$+Ck(k+1)(k+3)(2k+3)+Dk(k+1)(k+2)(2k+3)+$$

$$+Ek(k+1)(k+2)(k+3)$$

$k=-3$:	$k=-2$:	$k=-\frac{3}{2}$:	$k=-1$:	$k=0$:
$1=18D$	$1=-2C$	$1=\frac{9E}{16}$	$1=-2B$	$1=18A$
$D=\frac{1}{18}$	$C=-\frac{1}{2}$	$E=\frac{16}{9}$	$B=-\frac{1}{2}$	$A=\frac{1}{18}$

792. $0 < x_1 < 1, x_{n+1} = \sqrt{1+nx_n}$. **Find:**

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{kn^2}{n^4+k^2x_n^2}\right)$$

Proposed by Florică Anastase-Romania

Solution by Mokhtar Khassani-Mostaganem-Algerie

$$x_{n+1} = \sqrt{1+nx_n} \Rightarrow \frac{x_{n+1}^2}{n^2} = \frac{1}{n^2}+\frac{x_n}{n} \Rightarrow \lim_{n\to\infty}\frac{x_n}{n} = 1 \Rightarrow$$

$$\Rightarrow \Omega = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\frac{k}{n}}{1+\frac{k^2x_n^2}{n^4}} = \int_0^1\frac{x}{1+x^2}dx = \frac{\log 2}{2}$$

793. Find the closed form:

$$\Omega = \sum_{n=1}^{\infty}\frac{\left\{\frac{1}{n^4+n^2+1}+\frac{1}{8}\right\}}{n^2+1}, \{x\} = x-[x],$$

$[*]$ - great integer function

Proposed by Mokhtar Khassani-Mostaganem-Algerie

Solution 1 by Rohan Shinde-India

$\Omega = \sum_{n=1}^{\infty}\frac{\left\{\frac{1}{n^4+n^2+1}+\frac{1}{8}\right\}}{n^2+1}; \{x\} = x-[x] \to \forall n \geq 1$ *we have that*

$$n^4+n^2+1 \geq 3 \Rightarrow 0 \leq \frac{1}{n^4+n^2+1} \leq \frac{1}{3} \Rightarrow 0 \leq \frac{1}{n^4+n^2+1}+\frac{1}{8} \leq \frac{11}{24}$$

$\Rightarrow \left[\frac{1}{n^4+n^2+1}+\frac{1}{8}\right] = 0 \to [x]$ *denotes Greatest integer function/Floor function;* $\forall n \geq 1$

and $n \geq \mathbb{N} \Rightarrow \left\{\frac{1}{n^4+n^2+1}+\frac{1}{8}\right\} = \frac{1}{n^4+n^2+1}+\frac{1}{8}$

$$\Rightarrow \Omega = \sum_{n=1}^{\infty}\frac{\left(\frac{1}{n^4+n^2+1}+\frac{1}{8}\right)}{n^2+1} = \underbrace{\sum_{n=1}^{\infty}\frac{1}{(n^2+1)(n^4+n^2+1)}}_{A}+\underbrace{\frac{1}{8}\sum_{n=1}^{\infty}\frac{1}{n^2+1}}_{B}$$

Let's consider B first, using the Fourier series of $\coth(x)$ *we have that*

$\coth(x) = \frac{1}{x}+2\sum_{n=1}^{\infty}\frac{x}{n^2\pi^2+x^2}$. *Substitute $x = \pi$ to get*

$$\coth(\pi) = \frac{1}{\pi}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2+1} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2+1} = \frac{\pi\coth(\pi)-1}{2} = B$$

Now, let's consider A. Note that

$$2A = 2\sum_{n=1}^{\infty}\frac{1}{(n^2+1)(n^4+n^2+1)} = \left(\sum_{n=-\infty}^{\infty}\frac{1}{(n^2+1)(n^4+n^2+1)}\right)-1$$

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Let $F(z)=\frac{1}{(z^2+1)(z^4+z^2+1)}$***. By Cauchy Residue theorem,***

$$\sum_{n=-\infty}^{\infty}F(n)=-\sum(residues\ of\ \pi\cot(\pi z)\,F(z)\ at\ F's\ poles)$$

$F(z)$ ***has 6 simple poles namely at*** $i,-i,e^{\frac{i\pi}{3}},-e^{\frac{i\pi}{3}},e^{\frac{i2\pi}{3}},-e^{\frac{i2\pi}{3}}$***.***

$$Res_{z=i}(F(z))=-\frac{\pi}{2}\coth(\pi)$$

$$Res_{z=-i}(F(z))=-\frac{\pi}{2}\coth(\pi)$$

$$Res_{z=e^{\frac{i\pi}{3}}}(F(z))=\frac{\pi\tanh\left(\frac{\sqrt{3}\pi}{2}\right)}{\sqrt{3}-3i}$$

$$Res_{z=-e^{\frac{i\pi}{3}}}(F(z))=\frac{\pi\tanh\left(\frac{\sqrt{3}\pi}{2}\right)}{\sqrt{3}-3i}$$

$$Res_{z=e^{\frac{i2\pi}{3}}}(F(z))=\frac{\pi\tanh\left(\frac{\sqrt{3}\pi}{2}\right)}{\sqrt{3}+3i}$$

$$Res_{z=-e^{\frac{i2\pi}{3}}}(F(z))=\frac{\pi\tanh\left(\frac{\sqrt{3}\pi}{2}\right)}{\sqrt{3}+3i}$$

Combining all of the above we get:

$$\sum_{n=-\infty}^{\infty}F(n)=\pi\coth(\pi)-2\pi\tanh\left(\frac{\sqrt{3}\pi}{2}\right)\left(\frac{1}{\sqrt{3}-3i}+\frac{1}{\sqrt{3}+3i}\right)$$

$$=\pi\coth(\pi)-\frac{\pi}{\sqrt{3}}\tanh\left(\frac{\sqrt{3}\pi}{2}\right)\Rightarrow A=\frac{\pi}{2}\coth(\pi)-\frac{\pi}{2\sqrt{3}}\tanh\left(\frac{\sqrt{3}\pi}{2}\right)-\frac{1}{2}$$

$$\Rightarrow\Omega=A=\frac{1}{8}B\Rightarrow\Omega=\frac{\pi}{2}\coth(\pi)-\frac{\pi}{2\sqrt{3}}\tanh\left(\frac{\sqrt{3}\pi}{2}\right)-\frac{1}{2}+\frac{\pi\coth(\pi)}{16}-\frac{1}{16}$$

$=\frac{9\pi\coth(\pi)}{16}-\frac{\pi\tanh\left(\frac{\sqrt{3}\pi}{2}\right)}{2\sqrt{3}}-\frac{9}{16}$***. Answer:*** $\Omega=\frac{9\pi\coth(\pi)}{16}-\frac{\pi\tanh\left(\frac{\sqrt{3}\pi}{2}\right)}{2\sqrt{3}}-\frac{9}{16}$

Solution 2 by Ali Jaffal-Lebanon

$$\text{Let } S_n = \sum_{k=1}^{k=n} \frac{\left\{\frac{1}{k^4+k+1}+\frac{1}{8}\right\}}{k^2+1}$$

$$\text{If } 1 \le k \le n \text{ then } 3 \le k^4+k+1 \text{ and } 0 < \frac{1}{k^4+k+1}+\frac{1}{8} \le \frac{1}{3}+\frac{1}{8} < 1$$

$$\text{So, } \left[\frac{1}{k^4+k+1}+\frac{1}{8}\right] = 0 \text{ and } \left\{\frac{1}{k^4+k+1}+\frac{1}{8}\right\} = \frac{1}{k^4+k+1}+\frac{1}{8} - \left[\frac{1}{k^4+k+1}+\frac{1}{8}\right] = \frac{1}{k^2+k+1}+\frac{1}{8}$$

$$\text{We have: } \frac{\frac{1}{k^4+k+1}+\frac{1}{8}}{k^2+1} = \frac{1}{(k^2+1)(k^4+k+1)} + \frac{1}{8(k^2+1)}$$

$$\text{And by simple calculation we can say that: } \frac{1}{(k^2+1)(k^4+k+1)} =$$

$$= \frac{1}{k^2+1} - \frac{1}{2}\left(\frac{1}{k^2-k+1} + \frac{k-1}{(k-1)^2+(k-1)+1} - \frac{k}{k^2+k+1}\right)$$

$$\text{Then, } S_n = \frac{9}{8}\sum_{k=1}^{k=n}\frac{1}{k^2+1} - \frac{1}{2}\sum_{k=1}^{k=n}\frac{1}{k^2-k+1} + \frac{1}{2}\left(\frac{n}{n^2+n+1}\right)$$

$$\text{And } S_n = \frac{9}{8}\sum_{k=0}^{k=n}\frac{1}{k^2+1} - \frac{1}{2}\sum_{k=0}^{k=n}\frac{1}{k^2-k+1} + \frac{1}{2}\left(\frac{n}{n^2+n+1}\right) - \frac{5}{8} \quad (*)$$

$$\text{We have: } \sum_{n=0}^{+\infty}\frac{1}{1+n^2} = \sum_{n=0}^{+\infty}\frac{1}{(n-i)(n+i)} = \frac{\Psi(i)-\Psi(-i)}{i-(-i)} = \frac{\Psi(i)-\Psi(-i)}{2i}\ldots$$

$$\text{But } \frac{\Psi(i)-\Psi(-i)}{2i} = \frac{1}{2i}\left(\Psi(i) - \left(\Psi(i) + \pi\cot(\pi i) + \frac{1}{i}\right)\right)$$

$$\text{Since } \Psi(-z) = \left(\Psi(z) + \pi\cot(\pi z) + \frac{1}{z}\right) \text{ for all } z \in \mathbb{C}^* \quad (**)$$

$$\text{Then, } \sum_{n=0}^{+\infty}\frac{1}{1+n^2} = \frac{1}{2i}\left(\left(-\pi\cot(\pi i) + \frac{1}{i}\right)\right) = \frac{1}{2i}\left(-\pi\frac{e^{-\pi}+e^{\pi}}{e^{-\pi}-e^{\pi}}i - \frac{1}{i}\right) = \frac{\pi}{2}\coth(\pi) + \frac{1}{2}$$

$$\text{We have } \sum_{n=0}^{+\infty}\frac{1}{1+n+n^2} = \sum_{n=0}^{+\infty}\frac{1}{\left(n-\frac{1-i\sqrt{3}}{2}\right)\left(n-\frac{1+i\sqrt{3}}{2}\right)} = \frac{\Psi\left(\frac{-1+i\sqrt{3}}{2}\right)-\Psi\left(-\frac{1+i\sqrt{3}}{2}\right)}{\frac{-1+i\sqrt{3}}{2}+\frac{1+i\sqrt{3}}{2}}$$

$$\sum_{n=0}^{+\infty}\frac{1}{1+n+n^2} = \frac{1}{i\sqrt{3}}\left(\Psi\left(\frac{-1+i\sqrt{3}}{2}\right) - \Psi\left(-\frac{1+i\sqrt{3}}{2}\right)\right) \text{ by using (**) we have:}$$

$$= \frac{1}{i\sqrt{3}}\left(\Psi\left(\frac{-1}{2}+\frac{i\sqrt{3}}{2}\right) - \Psi\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right) - \pi\cot\left(\frac{\pi}{2}+\frac{i\sqrt{3}}{2}\right) - \frac{2}{1+i\sqrt{3}}\right)$$

$$= \frac{1}{i\sqrt{3}}\left(-\frac{1}{-\frac{1}{2}+\frac{i\sqrt{3}}{2}} - \pi\cot\left(\frac{\pi}{2}+\frac{i\sqrt{3}}{2}\right) - \frac{2}{1+i\sqrt{3}}\right). \text{ Since } \Psi(z) = \left(\Psi(z+1) - \frac{1}{z}\right)$$

$$\text{Then, } \cot\left(\frac{\pi}{2}+\frac{i\sqrt{3}}{2}\right) = \frac{ie^{-\frac{\pi}{2}\sqrt{3}}-ie^{\frac{\pi}{2}\sqrt{3}}}{ie^{-\frac{\pi}{2}\sqrt{3}}+ie^{\frac{\pi}{2}\sqrt{3}}} = i\tanh\left(\frac{\pi}{2}\sqrt{3}\right). \text{ So, } \sum_{n=0}^{+\infty}\frac{1}{1+n+n^2} = 1 + \frac{\pi}{\sqrt{3}}\tanh\left(\frac{\pi}{2}\sqrt{3}\right)$$

Then by using (*) we have:

$$\lim_{n\to+\infty} s_n = \frac{9}{8}\left(\frac{\pi}{2}\coth(\pi)+\frac{1}{2}\right)-\frac{1}{2}\left(1+\frac{\pi}{\sqrt{3}}\tanh\left(\frac{\pi}{2}\sqrt{3}\right)\right)-\frac{5}{8}$$

$$=\frac{9\pi}{16}\coth(\pi)-\frac{\pi}{2\sqrt{3}}\tanh\left(\frac{\pi}{2}\sqrt{3}\right)-\frac{9}{16}$$

794. $m,n,p\in\mathbb{N}, m,n,p\ge 2, a_i>0, a_i\neq 1, i\in\overline{1,p}$ **all fixed.**

Find:

$$\Omega=\lim_{x\to 0}\left(\left(\frac{a_1^{\sqrt[n]{x}}+a_2^{\sqrt[n]{x}}+\cdots+a_p^{\sqrt[n]{x}}}{a_1^{\sqrt[m]{x}}+a_2^{\sqrt[m]{x}}+\cdots+a_p^{\sqrt[m]{x}}}\right)^{\frac{1}{\sqrt[n]{x}+\sqrt[m]{x}}}\right)$$

Proposed by Florică Anastase-Romania

Solution by Igor Soposki-Skopje-Macedonia

$$\Omega=\lim_{x\to 0}\left(\frac{a_1^{\sqrt[n]{x}}+a_2^{\sqrt[n]{x}}+\cdots+a_p^{\sqrt[n]{x}}}{a_1^{\sqrt[m]{x}}+a_2^{\sqrt[m]{x}}+\cdots+a_p^{\sqrt[m]{x}}}\right)^{\frac{1}{\sqrt[n]{x}+\sqrt[m]{x}}}=$$

$$=\exp\left[\lim_{x\to 0}\left(\frac{a_1^{\sqrt[n]{x}}+a_2^{\sqrt[n]{x}}+\cdots+a_p^{\sqrt[n]{x}}}{a_1^{\sqrt[m]{x}}+a_2^{\sqrt[m]{x}}+\cdots+a_p^{\sqrt[m]{x}}}-1\right)\cdot\frac{1}{\sqrt[n]{x}+\sqrt[m]{x}}\right]=$$

$$=\exp\left[\lim_{x\to 0}\left(\frac{\left(a_1^{\sqrt[n]{x}}-a_1^{\sqrt[m]{x}}\right)+\left(a_2^{\sqrt[n]{x}}-a_2^{\sqrt[m]{x}}\right)+\cdots+\left(a_p^{\sqrt[n]{x}}-a_p^{\sqrt[m]{x}}\right)}{a_1^{\sqrt[m]{x}}+a_2^{\sqrt[m]{x}}+\cdots+a_p^{\sqrt[m]{x}}}\right)\cdot\frac{1}{\sqrt[n]{x}+\sqrt[m]{x}}\right]=$$

$$=\exp\left[\frac{1}{p}\cdot\left(\underbrace{\lim_{x\to 0}\frac{a_1^{\sqrt[n]{x}}-a_1^{\sqrt[m]{x}}}{\sqrt[n]{x}+\sqrt[m]{x}}}_{L_1}+\underbrace{\lim_{x\to 0}\frac{a_2^{\sqrt[n]{x}}-a_2^{\sqrt[m]{x}}}{\sqrt[n]{x}+\sqrt[m]{x}}}_{L_2}+\cdots+\underbrace{\lim_{x\to 0}\frac{a_p^{\sqrt[n]{x}}-a_p^{\sqrt[n]{x}}}{\sqrt[n]{x}+\sqrt[m]{x}}}_{L_p}\right)\right]=$$

$$\Rightarrow L_1=\lim_{x\to 0}\frac{a_1^{\sqrt[n]{x}}-a_1^{\sqrt[m]{x}}}{\sqrt[n]{x}+\sqrt[m]{x}}=\lim_{x\to 0}\frac{a_1^{\sqrt[n]{x}}-a_1^{\sqrt[m]{x}}}{\sqrt[n]{x}+\sqrt[m]{x}}\cdot\frac{a_1^{\sqrt[m]{x}}}{a_1^{\sqrt[m]{x}}}=$$

$$=\lim_{x\to 0}\frac{a_1^{\sqrt[n]{x}+\sqrt[m]{x}}-1-\left(a_1^{\sqrt[m]{x}}-1\right)\left(a_1^{\sqrt[m]{x}}+1\right)}{a_1^{\sqrt[m]{x}}\cdot\left(\sqrt[n]{x}+\sqrt[m]{x}\right)}=$$

$$= \lim_{x\to 0}\left(\frac{1}{a_1^{\sqrt[m]{x}}}\cdot\frac{a_1^{\sqrt[n]{x}+\sqrt[m]{x}}-1}{\sqrt[n]{x}+\sqrt[m]{x}}-\frac{a_1^{\sqrt[m]{x}}+1}{a_1^{\sqrt[m]{x}}}\cdot\frac{a_1^{\sqrt[m]{x}}-1}{\sqrt[m]{x}}\cdot\frac{\sqrt[m]{x}}{\sqrt[n]{x}+\sqrt[m]{x}}\right)=$$

$$= \lim_{x\to 0}\frac{1}{a_1^{\sqrt[n]{x}}}\cdot\lim_{x\to 0}\frac{a_1^{\sqrt[n]{x}+\sqrt[m]{x}}-1}{\sqrt[n]{x}+\sqrt[m]{x}}-\lim_{x\to 0}\frac{a_1^{\sqrt[m]{x}}+1}{a_1^{\sqrt[m]{x}}}\cdot\lim_{x\to 0}\frac{a_1^{\sqrt[m]{x}}-1}{\sqrt[m]{x}}\cdot\lim_{x\to 0}\frac{1}{1+\sqrt[m-n]{x^{(x-n)}}}=$$

$$= 1\cdot\ln a_1 - 2\cdot\ln a_1\cdot 1 = -\ln a_1\,; L_2 = -\ln a_2\,; L_p = -\ln a_p$$

$$\Omega = e^{\frac{1}{p}(L_1+L_2+\dots+L_p)} = e^{-\frac{1}{p}(\ln a_1+\ln a_2+\dots+\ln a_p)} = e^{-\ln(a_1a_2\dots a_p)^{\frac{1}{p}}} = \frac{1}{\sqrt[p]{a_1\cdot a_2\cdot\ldots\cdot a_p}}$$

795. **Let $\pi(x)$ denotes the prime counting function and p_n denotes the n^th^ prime number. Find:**

$$\Omega = \lim_{n\to\infty} p_n^{\frac{\pi(n)}{n}}$$

Proposed by Prem Kumar-India

Solution by Mokhtar Khassani-Mostaganem-Algerie

We know that: $\pi(n)\sim\frac{n}{\log n}\qquad P_n\sim n\log n$

$$\Omega = \lim_{n\to\infty}(P_n)^{\frac{\pi(n)}{n}} = \lim_{n\to\infty}(n\log n)^{\frac{1}{\log n}} = \lim_{n\to\infty} e^{\frac{1}{\log n}\log(n\log n)} = \lim_{n\to\infty} e^{\frac{\log n+\log(\log n)}{\log n}} = e$$

796. **Find:**

$$\Omega = \lim_{n\to\infty}\left(\binom{n}{3}\cdot\lim_{m\to\infty}\left(\frac{1}{m}\sum_{k=0}^{m}\left(\frac{\binom{m}{k}}{\binom{m+n}{k+3}}\right)\right)\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Mokhtar Khassani-Mostaganem-Algerie

Let $\omega = \sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{m+n}{k+3}} = \sum_{k=0}^{m}\binom{m}{n}\frac{(k+3)!(m+n-k-3)!}{(m+n)!} =$

$$= (n+m+1)\sum_{k=0}^{m}\binom{m}{k}\frac{\Gamma(k+4)\Gamma(m+n-k-2)}{\Gamma(m+n+2)}$$

$$= (n+m+1)\sum_{k=0}^{m}\binom{m}{k}\int_0^1 x^{k+3}(1-x)^{m+n-k-3}dx =$$

$$= (m+n+1)\int_0^1 x^3(1-x)^{n-3}\sum_{k=0}^{m}\binom{m}{k}x^k(1-x)^{m-k}dx = (n+m+1)\int_0^1 x^3(1-x)^{n-3}\,dx$$

$$= \frac{(n+m+1)\Gamma(4)\Gamma(n-2)}{\Gamma(n+2)} = \frac{6(m+n+1)}{(n-1)n(n+1)(n-2)} \Rightarrow$$

$$\Rightarrow \lim_{m\to\infty}\frac{\omega}{m} = \lim_{m\to\infty}\frac{6(m+n+1)}{m(n-1)n(n+1)(n-2)} = \frac{6}{(n-1)n(n+1)(n-2)}$$

Now: $\Omega = \lim_{n\to\infty}\left(\binom{n}{3}\lim_{m\to\infty}\frac{\omega}{m}\right) = \lim_{n\to\infty}\frac{n!}{6(n-3)!}\cdot\frac{6}{(n-1)n(n+1)(n-2)} = 0$

Solution 2 by Ali Jaffal-Lebanon

Let $\Omega(n,m) = \frac{1}{m}\sum_{k=0}^{k=m}\frac{C_m^k}{C_{m+n}^{k+3}} = \frac{1}{m}\sum_{k=0}^{k=n}C_m^k \times \frac{(k+3)!(m+n-k-3)!}{(m+n)!}$

$$= \frac{1}{m}\sum_{k=0}^{k=m}C_m^k \times \frac{\Gamma(k+1)\cdot\Gamma(m+n-k-2)}{\Gamma(m+n+2)} \times (m+n+1)$$

$$= \frac{m+n+1}{m}\sum_{k=0}^{k=m}C_m^k\int_0^1 x^{k+3}\cdot(1-n)^{m+n-k-3}dx = \frac{m+n+1}{m}\int_0^1\sum_{k=0}^{k=m}x^k(1-x)^{m-k}C_m^k x^3\cdot(1-x)^{n-3}dx$$

$$= \frac{m+n+1}{m}\int_0^1(x+1-x)^m\cdot x^3\cdot(1-x)^{n-3}dx = \frac{m+n+1}{m}\times\int_0^1 x^3(1-x)^{n-3}\,dx$$

$\lim_{n\to+\infty}\Omega(n.m) = \int_0^1 x^3(1-x)^{n-3}\,dx$. ***Let*** $I_n = C_n^3 \times \int_0^1 x^3(1-x)^{n-3}\,dx$

$$= C_n^3 \times \frac{\Gamma(4)\cdot\Gamma(n-2)}{\Gamma(n+2)} = \frac{n!}{(n-3)!\times 3!}\times\frac{3!\times(n-3)!}{(n+1)!} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$$

So, $\lim_{n\to\infty} I_n = 0$. ***Then*** $\Omega = 0$

Solution 3 by Tobi Joshua-Nigeria

$$\Omega = \lim_{n\to\infty}\left(C_3^n\left(\lim_{m\to\infty}\frac{1}{m}\sum_{k=0}^{m}\left(\frac{C_k^m}{C_{k+3}^{m+n}}\right)\right)\right)$$

$$\Omega=\lim_{n\to\infty}\left(C_3^n\left(\lim_{m\to\infty}\frac{1}{m}\sum_{k=0}^{m}\left(\frac{C_k^m(m+n-k-3)!\,(k+3)!}{(m+n)!}\right)\right)\right)$$

$$\Omega=\lim_{n\to\infty}\left(C_3^n\left(\lim_{m\to\infty}\frac{1}{m}\sum_{k=0}^{m}\left(\frac{C_k^m\Gamma(m+n-k-2)\Gamma(k+4)}{\Gamma(m+n+1)}\right)\right)\right)$$

$$\Omega=\lim_{n\to\infty}\left(C_3^n\left(\lim_{m\to\infty}\frac{1}{m}\sum_{k=0}^{m}\left(\frac{C_k^m\Gamma(m+n-k-2)\Gamma(k+4)}{\frac{\Gamma(m+n+2)}{(m+n+1)}}\right)\right)\right)$$

$$\Omega=\lim_{n\to\infty}\left(C_3^n\left(\lim_{m\to\infty}\sum_{k=0}^{m}\left(\frac{C_k^m\Gamma\left(\frac{2k+8}{2}\right)\Gamma\left(\frac{2m+2n-2k-4}{2}\right)}{\Gamma\left(\frac{2m+2n+4}{2}\right)}\cdot\frac{(m+n+1)}{m}\right)\right)\right)$$

$$\Omega=\lim_{n\to\infty}\left(C_3^n\left(\lim_{m\to\infty}\frac{(m+n+1)}{m}\sum_{k=0}^{m}\left(C_k^mB(2k+7,2m+2n-2k-5)\right)\right)\right)$$

Recall that $\int_0^{\frac{\pi}{2}}\sin^a x\cos^b x\,dx=\frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}{2\Gamma\left(\frac{a+b+2}{2}\right)}$

$$\Omega=\lim_{n\to\infty}\left(C_3^n\left(\lim_{m\to\infty}\frac{(m+n+1)}{m}\sum_{k=0}^{m}\left(2C_k^m\int_0^{\frac{\pi}{2}}\sin^{(2k+7)}x\cos^{(2m+2n-2k)}x\,dx\right)\right)\right)$$

$$\Omega=\lim_{n\to\infty}\left(C_3^n\left(\lim_{m\to\infty}\frac{(m+n+1)}{m}\sum_{k=0}^{m}\left(2C_k^m\frac{\sin^7 x\cos^{2n}x}{\cos^5 x}\int_0^{\frac{\pi}{2}}\sin^{(2k)}x\cos^{(2m)}x\,dx\right)\right)\right)$$

$$\Omega=\lim_{n\to\infty}\left(C_3^n\left(\lim_{m\to\infty}2\frac{(m+n+1)}{m}\sum_{k=0}^{m}C_k^m\sin^{(2k)}x\cos^{(2m)}x\times\int_0^{\frac{\pi}{2}}\frac{\sin^7 x\cos^{2n}x}{\cos^5 x}dx\right)\right)$$

$$\Omega = \lim_{n\to\infty}\left(C_3^n\left(\lim_{m\to\infty}\frac{2(m+n+1)}{m}(\sin^2 x+\cos^2 x)^m \times \int_0^{\frac{\pi}{2}}\frac{\sin^7 x\cos^{2n} x}{\cos^5 x}dx\right)\right)$$

$$\Omega = \lim_{n\to\infty}\left(C_3^n\left(\lim_{m\to\infty}\frac{(m+n+1)}{m}2\int_0^{\frac{\pi}{2}}\sin^7 x\cos^{(2n-5)} x\,dx\right)\right)$$

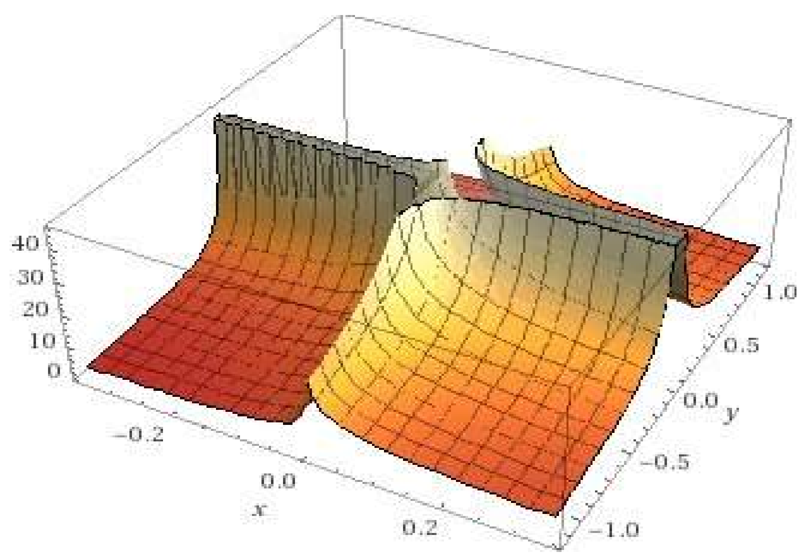
$$\Omega = \lim_{n\to\infty}\left(C_3^n\left(\lim_{m\to\infty}\frac{(m+n+1)}{m}\frac{\Gamma(4)\Gamma(n-2)}{\Gamma(n+2)}\right)\right),$$

$$\Omega = \lim_{n\to\infty}\left(C_3^n\left(\lim_{m\to\infty}\frac{(m+n+1)}{m}\frac{3!\times(n-3)!}{(n+1)!}\right)\right)$$

$$\Omega = \lim_{n\to\infty}\left(C_3^n\left(\lim_{m\to\infty}\left(1+\frac{n}{m}+\frac{1}{m}\right)\frac{3!\times(n-3)!}{(n+1)!}\right)\right) = \lim_{n\to\infty}\left(\frac{n!}{(n-3)!\,3!}\times(1)\times\frac{3!\times(n-3)!}{(n+1)!}\right)$$

$$\Omega = \lim_{n\to\infty}\frac{(n)!}{(n+1)!},\Omega = \lim_{n\to\infty}\frac{1}{(n+1)},\Omega = \lim_{n\to\infty}\frac{\frac{1}{n}}{\left(1+\frac{1}{n}\right)} = 0$$

797. Find the point on the surface $xy^2z^3 = 108$ that are closest to the origin.



Proposed by Jalil Hajimir-Canada

Graphics by Carlos Suarez-Quito-Ecuador

Solution by Daniel Sitaru-Romania

$$F(x,y,z,\lambda) = x^2 + y^2 + z^2 + \lambda(xy^2z^3 - 108), x \neq 0, y \neq 0, z \neq 0$$

$$F'_x = 2x + \lambda y^2 z^3, F'_y = 2y(1 + \lambda xz^3) = 0 \to \lambda = \frac{-1}{xz^3}, F'_z = z(2 + 3xy^2 z\lambda)$$

$$2x + \frac{-1}{xz^3} \cdot y^2 z^3 = 0 \to 2x^2 = y^2 \to y = x\sqrt{2}$$

$$2 + 3xy^2 z \cdot \frac{-1}{xz^3} = 0 \to 2z^2 = 3 \cdot 2x^2 \to z = x\sqrt{3}$$

$$x \cdot 2x^2 \cdot 3\sqrt{3}x^3 = 108 \to \sqrt{3}x^6 = 18 \to x^6 = 6\sqrt{3} = \sqrt{108}$$

$$x = \sqrt[12]{108}, y = \sqrt[12]{108 \cdot 2^6}, z = \sqrt[12]{108 \cdot 3^6}$$

The closest point is $\Omega\left(\sqrt[12]{108}, \sqrt[12]{108 \cdot 2^6}, \sqrt[12]{108 \cdot 3^6}\right)$

798. Solve for real numbers:

$$\begin{cases} \sin x = \cos y \\ \begin{vmatrix} \sin(x+y) & \sin(y+\sqrt{xy}) & \sin(\sqrt{xy}+x) \\ \cos(x+y) & \cos(y+\sqrt{xy}) & \cos(\sqrt{xy}+x) \\ \cos(x-y) & \cos(y-\sqrt{xy}) & \cos(\sqrt{xy}-x) \end{vmatrix} = 0 \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution by Adrian Popa – Romania

$$\Delta \to \begin{vmatrix} \sin(x+y) & \sin(y+\sqrt{xy}) & \sin(\sqrt{xy}+x) \\ \cos(x+y) & \cos(y+\sqrt{xy}) & \cos(\sqrt{xy}+x) \\ \cos(x-y) & \cos(y-\sqrt{xy}) & \cos(\sqrt{xy}-x) \end{vmatrix} = 0$$

$$\sin x = \cos y \Rightarrow \sin x = \sin\left(\frac{\pi}{2} - y\right)$$

$$\sin(x+y) = \sin x \cos y + \sin y \cos x = \cos^2 y + \sin^2 y = 1 \quad \textit{(1)}$$

$$\therefore \sin^2 x = \cos^2 y$$

$$1 - \cos^2 x = \cos^2 y \Rightarrow \cos^2 x = 1 - \cos^2 y \Rightarrow \cos x = \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y = \sin y \cos y - \cos y \sin y = 0 \quad \textit{(2)}$$

$$\cos(x-y)=\cos x\cos y+\sin x\sin y=\sin y\cos y+\cos y\sin y=2\sin y\cos y=\sin 2y \quad \textit{(3)}$$

$$\Delta=\begin{vmatrix} 1 & \sin\left(y+\sqrt{xy}\right) & \sin\left(\sqrt{xy}+x\right) \\ 0 & \cos\left(y+\sqrt{xy}\right) & \cos\left(\sqrt{xy}+x\right) \\ \sin 2y & \cos\left(y-\sqrt{xy}\right) & \cos\left(\sqrt{xy}-x\right) \end{vmatrix}=0$$

We develop after the first column:

$$\cos\left(y+\sqrt{xy}\right)\cdot\cos\left(\sqrt{xy}-x\right)-\cos\left(\sqrt{xy}+x\right)\cdot\cos\left(y-\sqrt{xy}\right)+$$

$$+\sin 2y\left(\sin\left(y+\sqrt{xy}\right)\cdot\cos\left(\sqrt{xy}+x\right)-\sin\left(\sqrt{xy}+x\right)\cdot\cos\left(y+\sqrt{xy}\right)\right)=0\Rightarrow$$

$$\Rightarrow\cos\left(y+\sqrt{xy}\right)\cdot\cos\left(\sqrt{xy}-\frac{\pi}{2}+y\right)-\cos\left(\sqrt{xy}+\frac{\pi}{2}-y\right)\cdot\cos\left(y-\sqrt{xy}\right)+$$

$$+\sin 2y\left(\sin\left(y+\sqrt{xy}\right)\cdot\cos\left(\sqrt{xy}+\frac{\pi}{2}-y\right)-\sin\left(\sqrt{xy}+\frac{\pi}{2}-y\right)\cdot\cos\left(y+\sqrt{xy}\right)\right)=0$$

$$\cos\left(y+\sqrt{xy}\right)\cdot\sin\left(y+\sqrt{xy}\right)-\sin\left(y-\sqrt{xy}\right)\cdot\cos\left(y-\sqrt{xy}\right)+$$

$$+\sin 2y\left(\sin\left(y+\sqrt{xy}\right)\cdot\sin\left(y-\sqrt{xy}\right)-\cos\left(y-\sqrt{xy}\right)\cdot\cos\left(y+\sqrt{xy}\right)\right)=0\Rightarrow$$

$$\Rightarrow\frac{\sin 2\left(y+\sqrt{xy}\right)}{2}-\frac{\sin 2\left(y-\sqrt{xy}\right)}{2}-\sin 2y\left(\cos 2y\right)=0\Rightarrow$$

$$\Rightarrow\frac{2\sin 2\sqrt{xy}\cos 2y}{2}-\sin 2y\cos 2y=0$$

$$\cos 2y\left(\sin 2\sqrt{xy}-\sin 2y\right)=0$$

Case I: $\cos 2y=0\Rightarrow 2y=\pm\frac{\pi}{2}+2k\pi\Rightarrow\begin{cases} y=\pm\frac{\pi}{4}+k\pi\Rightarrow \\ x=\frac{\pi}{2}-y=\frac{\pi}{2}\mp\frac{\pi}{4}-k\pi\end{cases};k\in\mathbb{Z}$

Case II: $\sin 2\sqrt{xy}-\sin 2y=0\Rightarrow\sin 2\sqrt{xy}=\sin 2y\Rightarrow\sin 2\sqrt{xy}=\sin 2\sqrt{y\cdot y}\Rightarrow$

$$\Rightarrow\begin{cases} y=0 \\ x=\frac{\pi}{2}+2k\pi\end{cases};k\in\mathbb{N},\ \textbf{\textit{because}}\ x\neq y;\ \begin{matrix}\sin x>0\\ y>0\end{matrix}$$

799. Solve in real numbers the system of equations:

$$\begin{cases} x+y+z+t=0, \\ x^2+y^2+z^2+t^2=4, \\ x^4+y^4+z^4+t^4=4.\end{cases}$$

Proposed by Hung Nguyen Viet-Hanoi-Vietnam

Solution 1 by Daniel Văcaru-Romania

One has $16 = 4^2 = (x^2 + y^2 + z^2 + z^2)^2 = x^4 + y^4 + z^4 + t^4 + 2\sum x^2y^2 \Rightarrow$

$\Rightarrow \sum x^2y^2 = 6$. *On the other hand, one has* $x^4 + y^4 \ge 2x^2y^2$, $x^4 + z^4 \ge 2x^2z^2$,

$x^4 + t^4 \ge 2x^2t^2$, $y^4 + z^4 \ge 2y^2z^2$, $y^4 + t^4 \ge 2y^2t^2$, $z^4 + t^4 \ge 2z^2t^2$. *Summing this inequality, one obtains* $3(x^4 + y^4 + z^4 + t^4) \ge 2\sum x^2y^2$.

But we obtain $12 = 3 \cdot 4 = 3(x^4 + y^4 + z^4 + t^4) = 2\sum x^2y^2 = 2 \cdot 6 = 12$, *and this is followed by* $x^2 = y^2 = z^2 = t^2 = 1$. *We obtain:*

$$(x, y, z, t) \in \{(1,1,-1,-1), (1,-1,1,-1), (1,-1,-1,1), (-1,-1,1,1)\}$$
$$\cup \{(-1,1,-1,1), (-1,1,1,-1)\}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum x \overset{(a)}{=} 0, \sum x^2 \overset{(b)}{=} 4, \sum x^4 \overset{(c)}{=} 4$$

Let $x + y + z = \sigma(x)$, $x^2 + y^2 + z^2 = \sigma(x^2)$ *and* $x^4 + y^4 + z^4 = \sigma(x^4)$

Now, $\sigma(x^4) \overset{(1)}{=} 4 - t^4$ *and* $\sigma(x^2) \overset{(2)}{=} 4 - t^2 \because \sigma(x^4) \ge \frac{1}{3}\{\sigma(x^2)\}^2$

$\therefore 4 - t^2 \ge \frac{1}{3}(4 - t^2)^2$ *(using (1), (2))* $\Rightarrow 12 - 3t^4 \ge 16 - 8t^2 + t^4$

$$\Rightarrow 4t^4 - 8t^2 + 4 \le 0 \Rightarrow 4(t^2 - 1)^2 \le 0$$

$$\Rightarrow t^2 - 1 = 0 (\because (t^2 - 1)^2 \ge 0 \; and \; (t^2 - 1)^2 \le 0) \Rightarrow t^2 = t^4 = 1$$

$\therefore \sigma(x^2) \underset{by\,(b)}{\overset{(3)}{=}} 3$ *and* $\sigma(x^4) \overset{by\,(c)}{=} 3$ *(using (1), (2))*

Again, $\sigma(x^4) \ge \frac{1}{3}\left(\sigma(x^2)\right)^2 \overset{by\,(3)}{=} (\because \sigma(x^2) = 3)$

$\Rightarrow \sigma(x^4) \ge 3$ *with equality at* $x = y = z$ *and* $\because \sigma(x^4) = 3 \therefore$ *equality occurs*

$\Rightarrow x = y = z$. *Putting* $x = y = z$ *in (3), we get* $3x^2 = 3$

$\Rightarrow x^2 = y^2 = z^2 = t^2 = 1$ *and* $\because \sum x = 0$ *all possible solutions are:*

$$\begin{pmatrix} x = 1 \\ y = 1 \\ z = -1 \\ t = -1 \end{pmatrix}, \begin{pmatrix} x = 1 \\ y = -1 \\ z = -1 \\ t = -1 \end{pmatrix}, \begin{pmatrix} x = 1 \\ y = -1 \\ z = -1 \\ t = -1 \end{pmatrix}, \begin{pmatrix} x = -1 \\ y = 1 \\ z = 1 \\ t = -1 \end{pmatrix}, \begin{pmatrix} x = -1 \\ y = 1 \\ z = -1 \\ t = 1 \end{pmatrix}, \begin{pmatrix} x = -1 \\ y = -1 \\ z = 1 \\ t = 1 \end{pmatrix} \textit{(Answer)}$$

Solution 3 by Khaled Abd Imouti-Damascus-Syria

$$x + y + z + t = 0 \quad (1)$$

$$x^2 + y^2 + z^2 + t^2 = 4 \quad \textbf{(2)}$$

Plane: $P: x + y = z = -t$. ***Sphere:*** $S: x^2 + y^2 + z^2 = 4 - t^2$

$$M_0(0,0,0), R = \sqrt{4 - t^2}. \text{ Dis } (M_0, P) = \frac{|t|}{\sqrt{3}}$$

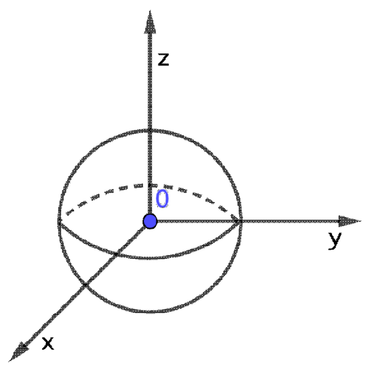
$$\frac{|t|}{\sqrt{3}} \le \sqrt{4 - t^2};\ t^2 \le 3(4 - t^2);\ 4t^2 \le 12;\ t^2 \le 13;\ -\sqrt{3} \le t \le +\sqrt{3}$$

There are six solutions: $P \cap S = \left\{ \begin{matrix} (1,1,-1,-1) \\ (-1,1,1,-1) \\ (-1,-1,1,1) \\ (1,-1,1,-1) \\ (-1,+1,-1,+1) \\ (-1,1,1,-1) \\ (1,-1,-1,1) \end{matrix} \right\}$

These six solutions are satisfying the equation: $x^4 + y^4 + z^4 + t^4 = 4$.

$$x^2 + y^2 + z^2 + t^2 - 3 = 1;\ t^2 - 3 \le 0;\ x^2 + y^2 + z^2 + t^2 - 3 \le x^2 + y^2 + z^2$$

$1 \le x^2 + y^2 + z^2$. ***So, plan must be tangent*** s_p ***have*** S'



$$S = \{1, -1\};\ C_4^2 = \frac{4!}{2! \cdot 2!} = 6$$

x	y	z	t		
$+1$	$+1$	$+1$	1	$\to$	4
$+1$	$+1$	$+1$	-1	$\to$	2
$+1$	$+1$	-1	1	$\to$	2
$+1$	$+1$	-1	-1	$\to$	0
$+1$	-1	1	1	$\to$	2

$+1$	-1	1	-1	$\rightarrow$	0
$+1$	-1	-1	1	$\rightarrow$	0
$+1$	-1	-1	-1	$\rightarrow$	-2
-1	$+1$	1	1	$\rightarrow$	2
-1	$+1$	1	-1	$\rightarrow$	0
-1	$+1$	-1	1	$\rightarrow$	0
-1	$+1$	-1	-1	$\rightarrow$	-2
-1	-1	1	1	$\rightarrow$	0
-1	-1	1	-1	$\rightarrow$	-2
-1	-1	-1	1	$\rightarrow$	-2
-1	-1	-1	-1	$\rightarrow$	-4

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

$$x+y+z+t=0 \quad (1)$$

$$x^2+y^2+z^2+t^2=4 \quad (2)$$

$$x^4+y^4+z^4+t^4=4 \quad (3)$$

From (1), $x+y=-(z+t) \Rightarrow x^2+y^2+2xy=z^2+t^2+2zt$

$$\Rightarrow 2xy=2(z^2+t^2)+2zt-4$$

$$\Rightarrow xy=z^2+t^2+zt-2 \quad (3)$$

From (3), $x^2+y^2=4-(z^2+t^2) \Rightarrow x^4+y^4+2(xy)^2=$

$$=16+z^4+t^4+2(zt)^2=8(z^2+t^2)$$

$$\Rightarrow 2(xy)^2=2(z^4+t^4)+2(zt)^2-8(z^2+t^2)+12$$

$$\Rightarrow (xy)^2=z^4+t^4+(zt)^2-4(z^2+t^2)+6 \quad (4)$$

From (3) and (4), we have $zt(z^2+t^2+zt-2)=1$

and since $x,y,z,t\in\mathbb{R}$*, hence* $x,y,z,t\in\mathbb{Z}$

1. If $zt=1$*, then* $(z^2+t^2+zt-2)=1 \Rightarrow z^2+t^2\le 2$

2. If $zt=-1$*, then* $z^2+t^2+zt-2=-1 \Rightarrow z^2+t^2=2$

$\Rightarrow z=1, t=1$ *or* $z=-1, t=1$

and since $x+y+z+t=0$*, hence its solution is*

$$\{(1,1,-1,-1),(1,-1,1,-1),(1,-1,-1,1),(-1,1,1,1),(-1,1-1,1),(-1,1,1,-1)\}$$

800. **Solve for real numbers:**

$$\begin{cases} 1 \le x, y, z \le 3 \\ (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = \frac{35}{3} \\ 3^y + \log_2 z = 3 \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Khaled Abd Imouti-Damascus-Syria

Solve for real numbers: $S' \begin{cases} 1 \le x, y, z \le 3 \\ (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = \frac{35}{3} \\ 3^y + \log_2(z) = 3 \end{cases}$

$$x \in [1,3], y \in [1,3], z \in [1,3]$$

$$S \begin{cases} (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = \frac{3S}{3} \\ 3^y + \frac{\ln(z)}{\ln(2)} = 3 \end{cases}$$

$$3^y = 3 - \frac{1}{\ln(2)} \cdot \ln(z), 1 \le z \le 3; f(z) = 3^y$$

$$f(3) = 3 - \frac{1}{\ln(2)} \cdot \ln(z), [1,3];\ f(1) = 3, f(3) = 3 - \frac{\ln(3)}{\ln(2)};\ f'(z) = -\frac{1}{\ln(2)} \cdot \frac{1}{z} < 0$$

z	1	3
$f'(z)$	− − − − − − − − − − − − − − − − − − −	
$f(z)$	3 ↘	$3 - \frac{\ln(3)}{\ln(2)}$

But: $1 \le y \le 3, g(y) = 3^y = e^{y\ln(3)}$

$$g(1) = 3, g(3) = 27, g'(y) = \ln(3) \cdot e^{y\ln(3)} > 0$$

y	1	3
$g'(y)$	+ + + + + + + + + + + + + + + + + +	
$g(y)$	3 ↗	27

$f(z) = g(y)$ *when:* $y = 1$ *and* $z = 1$

$$(x+2)\left(\frac{1}{x}+2\right) = \frac{35}{3} \Rightarrow (x+2)\left(\frac{1+2x}{x}\right) = \frac{35}{3}$$

$$(x+2)(1+2x)=\frac{35}{3}x \Rightarrow 3(x+2)(1+2x)=35x$$

$$3(x+2x^2+2+4x)=35x \Rightarrow 3(2x^2+5x+2)=35x$$

$$6x^2+15x-35x+6=0 \Rightarrow 6x^2-20x+6=0 \div (2) \Rightarrow 3x^2-10x+3=0$$

$$\Delta=100-4(3)(3)=100-36=64 \Rightarrow \sqrt{\Delta}=8$$

$$x=\frac{10-8}{6}=\frac{2}{6}=\frac{1}{3}\notin[1,3] \textit{ or } x=\frac{10+8}{6}=\frac{18}{6}=3\in[1,3)$$

$$\textit{So: } (x,y,z)=(3,1,1)$$

Solution 2 by Lazaros Zachariadis-Thesaloniki-Greece

$$\textit{if } z\in(1,3] \textit{ then } \frac{\ln z}{\ln 2}>0 \Leftrightarrow \log_2 z>0$$

$$\textit{if } y\in(1,3] \textit{ then } 3^y>3 \quad (+)$$

$$\overline{\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$\textit{So: } 3^y+\log_2 z>3$$

$$\textit{Likewise, if } z\in[1,3]\vee y\in(1,3] \textit{ and } z\in(1,3]\vee y\in[1,3]$$

$$3^y+\log_2 z=3 \textit{ if-f } y=z=1$$

$$\textit{Then } (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=\frac{35}{3} \textit{ because } (x+2)\left(\frac{1}{x}+2\right)=\frac{35}{3}$$

$$1+2x+\frac{2}{x}+4=\frac{35}{3} \Leftrightarrow 2\left(x+\frac{1}{x}\right)=\frac{\infty}{3} \Leftrightarrow x+\frac{1}{x}=\frac{10}{3} \overset{3x}{\Leftrightarrow}$$

$$\Leftrightarrow 3x^2-10x+3=0$$

$$x=\frac{10\pm\sqrt{100-4\cdot 9}}{2\cdot 3}=\frac{10\pm 8}{6}\begin{cases}\frac{18}{6}=3\\ \frac{2}{6}=\frac{1}{3}\end{cases}$$

$$(x,y,z)=(3,1,1)$$

Solution 3 by Ruangkhaw Chaoka-Chiangrai-Thailand

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=\frac{35}{3} \quad \textit{(1)}$$

$$3^y+\log_2 z=3 \quad \textit{(2)}$$

$$\textit{We'll show that } (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\le\frac{35}{3}; 1\le x,y,z\le 3$$

$$\textit{Or } \frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\overset{??}{\le}\frac{26}{3}$$

WLOG $x \geq y \geq z \Rightarrow 0 \leq (x-y)(y-z) \Leftrightarrow zx + y^2 \leq xy + yz$

$$\begin{cases} \frac{z}{y} + \frac{y}{x} \leq 1 + \frac{z}{x} \\ \frac{x}{y} + \frac{y}{z} \leq \frac{x}{z} + 1 \end{cases} \overset{+}{\Rightarrow} \frac{x}{y} + \frac{y}{z} + \frac{z}{y} + \frac{y}{x} \leq \frac{x}{z} + \frac{z}{x} + 2$$

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + \frac{y}{x} + \frac{z}{y} + \frac{x}{z} \leq 2\left(\frac{x}{z} + \frac{z}{x}\right) + 2$$

It remains to show that $2\left(\frac{x}{z} + \frac{z}{x}\right) + 2 \leq \frac{26}{3}$ *(3)*

Let $t = \frac{x}{z}; 1 \leq t \leq 3 \Leftrightarrow$ *(3);* $t + \frac{1}{t} \leq \frac{10}{3} \Leftrightarrow (3t-1)(t-3) \leq 0$

Which is true $\therefore$ *(1) is the hold point of inequality at* $t = \frac{x}{z} = 3$

$\begin{cases} x = 3, z = 1 \Rightarrow y = 1 \\ x = 3, z = 1 \Rightarrow y = 3 \end{cases}$ *and permutations* $\rightarrow$ *(2)*

$$\therefore (x, y, z) = (3, 1, 1)$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru