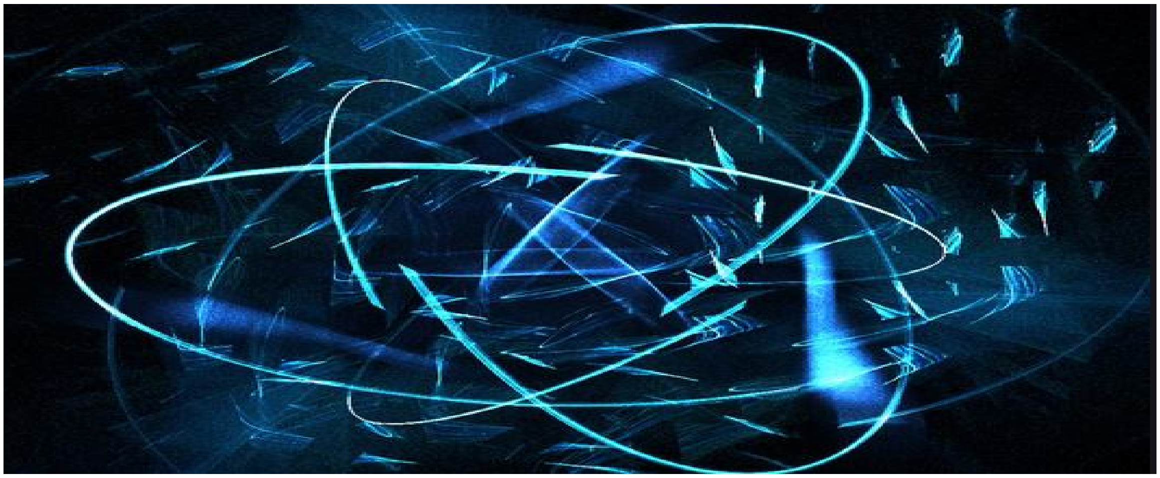
R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro



Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{n(n+1)}\sum_{k=1}^{n}\left(k\tan^{-1}\left(\frac{k^2+k}{n^2+n}\right)\right)\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ali Jaffal-Lebanon, Solution 2 by Remus Florin Stanca-Romania, Solution 3 by Marian Ursărescu-Romania

Solution 1 by Ali Jaffal-Lebanon

Let $f(x) = 1 - \frac{1}{x+1}, x \in [0, +\infty[$

$f'(x) = \frac{1}{(x+1)^2} > 0$ *then* f *is increasing on* $[0, +\infty[$

Let $n \in \mathbb{N}^*$ *and* $1 \le k \le n$, *so,* $f(k) \le f(n)$, *then* $1 - \frac{1}{k+1} \le 1 - \frac{1}{n+1}$

then $\frac{k}{k+1} \le \frac{n}{n+1}; \frac{k}{n} \le \frac{k+1}{n+1}$

then $\frac{k(k+1)}{n(n+1)} \le \frac{(k+1)^2}{(n+1)^2}$ *(*)*

but $\frac{k}{k+1} \le \frac{n}{n+1} \Rightarrow \frac{k^2}{k^2+k} \le \frac{n^2}{(n+1)n}$

So, $\frac{k^2}{n^2} \le \frac{k^2+k}{n^2+n}$ *(**)*

Then by (*); (**) we have: $\frac{k^2}{n^2} \le \frac{k(k+1)}{n(n+1)} \le \frac{(k+1)^2}{(n+1)^2}$

but $x \to \arctan x$ *is increasing, so:* $\arctan\left(\frac{k^2}{n^2}\right) \le \arctan\left(\frac{k(k+1)}{n(n+1)}\right) \le \arctan\left(\left(\frac{k+1}{n+1}\right)^2\right)$

$$\text{So, } \frac{1}{n(n+1)}\sum_{k=1}^{k=n} k\arctan\left(\frac{k}{n}\right)^2 \le S_n \le \frac{1}{n(n+1)}\sum_{k=1}^{k=n}(1+k)\arctan\left(\left(\frac{k+1}{n+1}\right)^2\right)$$

$$\text{Where } S_n = \frac{1}{n(n+1)}\sum_{k=1}^{k=n} k\arctan\left(\frac{k(k+1)}{n(n+1)}\right)$$

$$\text{We have } \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{k=n}\frac{k}{n}\arctan\left(\left(\frac{k}{n}\right)^2\right) = \int_0^1 x\arctan(x^2)\,dx$$

$$\text{Let } u = \arctan(x^2);\ u' = \frac{2x}{1+x^4};\ v' = x;\ v = \frac{x^2}{2}$$

$$\text{then } \int_0^1 x\arctan(x^2)\,dx = \frac{\pi}{8} - \int_0^1 \frac{x^3}{1+x^4}dx = \frac{\pi}{8} - \frac{1}{4}[\ln(1+x^4)]_0^1 = \frac{\pi}{8} - \frac{1}{4}\ln 2$$

$$\text{We have } \lim_{n\to\infty}\frac{1}{n(n+1)}\sum_{k=1}^{k=n} k\arctan\left(\left(\frac{k}{n}\right)^2\right) = \frac{\pi}{8} - \frac{1}{4}\ln 2$$

$$\text{and } \lim_{n\to\infty}\frac{1}{n(n+1)}\sum_{k=1}^{k=n}(1+k)\arctan\left(\left(\frac{k+1}{n+1}\right)^2\right) =$$

$$\lim_{n\to\infty}\frac{1}{n(n+1)}\sum_{t=2}^{t=n+1} t\arctan\left(\left(\frac{t}{n+1}\right)^2\right) = \frac{\pi}{8} - \frac{1}{4}\ln 2$$

$$\text{Then } \lim_{n\to\infty} S_n = \frac{\pi}{8} - \frac{1}{4}\ln 2$$

Solution 2 by Remus Florin Stanca-Romania

We know that $\lim_{n\to\infty}\sum_{k=1}^{n}(x_{k+1}-x_k)f(\zeta_k) = \int_a^b f(x)\,dx$, f *= continuous and* $\zeta_k \in [x_k; x_{k+1}]$ *and* $\lim_{n\to\infty}||\Delta_n|| = 0$ *where* $||\Delta_n|| \overset{\max}{\underset{k\le n}{=}} (x_{k+1}-x_k)$*, and* $x_1 = a$ *and* $x_n = b$*, let* $x_k = \frac{k(k+1)}{n(n+1)} \Rightarrow x_{k+1} - x_k = \frac{(k+1)(k+2)}{n(n+1)} - \frac{k(k+1)}{n(n+1)} = \frac{k^2+2k+k+2-k^2-k}{(n+1)} =$

$= 2\cdot\frac{k+1}{n(n+1)} \overset{k\le n}{\Rightarrow} \max(x_{k+1}-x_k) = 2\cdot\frac{n+1}{n(n+1)} = \frac{2}{n}$ *and* $\lim_{n\to\infty}\max(x_{k+1}-x_k) = 0 \Rightarrow$

$\Rightarrow \lim_{n\to\infty}||\Delta_n|| = 0$ *and* $\zeta_k = \frac{k(k+1)}{n(n+1)}$ *and* $f(x) = \tan^{-1}(x)$ *= continuous*

$$\Rightarrow \lim_{n\to\infty}\sum_{k=1}^{n}\frac{2(k+1)}{n(n+1)}\tan^{-1}\left(\frac{k^2+k}{n^2+n}\right) = \int_0^1 \tan^{-1}(x)\,dx \Rightarrow$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n(n+1)}\tan^{-1}\left(\frac{k^2+k}{n^2+n}\right) = \frac{1}{2}\int_0^1 \tan^{-1}(x)\,dx - \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n(n+1)}\tan^{-1}\left(\frac{k^2+k}{n^2+n}\right)$$

(1)

$$\tan^{-1}(x) = \textit{increasing} \Rightarrow \tan^{-1}\left(\frac{k^2+k}{n^2+n}\right) \le \tan^{-1}\left(\frac{n^2+n}{n^2+n}\right) = \frac{\pi}{4} \Rightarrow$$

$$\Rightarrow \frac{1}{n(n+1)}\sum_{k=1}^{n}\tan^{-1}\left(\frac{k^2+k}{n^2+n}\right) \le \frac{1}{n(n+1)}\sum_{k=1}^{n}\frac{\pi}{4} = \frac{\frac{\pi}{4}}{n+1} \textit{ and}$$

$$\lim_{n\to\infty}\frac{\frac{\pi}{4}}{n+1} = 0 \Rightarrow \lim_{n\to\infty}\frac{1}{n(n+1)}\sum_{k=1}^{n}\tan^{-1}\left(\frac{k^2+n}{n^2+n}\right) = 0 \overset{(1)}{\Rightarrow}$$

$$\overset{(1)}{\Rightarrow} \lim_{n\to\infty}\left(\frac{1}{n(n+1)}\sum_{k=1}^{n}\left(k\tan^{-1}\left(\frac{k^2+k}{n^2+n}\right)\right)\right) = \frac{1}{2}\int_0^1 \tan^{-1}(x)\,dx \quad \textit{(2)}$$

$$\int \tan^{-1}(x)\,dx = \int \tan^{-1}(x)\cdot x'\,dx = x\tan^{-1}(x) - \int \frac{x}{x^2+1}dx =$$

$$= x\tan^{-1}(x) - \frac{1}{2}\int\frac{2x}{x^2+1} = x\tan^{-1}(x) - \frac{1}{2}\ln(x^2+1) \overset{(2)}{\Rightarrow}$$

$$\overset{(2)}{\Rightarrow} \lim_{n\to\infty}\left(\frac{1}{n(n+1)}\sum_{k=1}^{n}\left(k\tan^{-1}\left(\frac{k^2+k}{n^2+n}\right)\right)\right) = \frac{1}{2}\left(\frac{\pi}{4} - \frac{\ln(2)}{2}\right) =$$

$$= \frac{\pi}{8} - \frac{\ln(2)}{4} = \frac{\pi - \ln(4)}{8} \Rightarrow \Omega = \frac{\pi - \ln(4)}{8}$$

Solution 3 by Marian Ursărescu-Romania

(another approach)

Let $f:[0,1]\to\mathbb{R}, f(x) = \arctan x;\ f -$ *Riemann integrability*

Let $\Delta_n = \left(0, \frac{1-2}{n(n+1)}, \dots \frac{k(k+1)}{n(n+1)}, \dots \frac{n(n+1)}{n(n+1)} = 1\right)$

$$||\Delta_n|| = \frac{1}{n(n+1)}\max_{1\le k\le n}(k(k+1) - (k-1)(k)) = \frac{1}{n(n+1)}\cdot 2k = \frac{2n}{n(n+1)} \Rightarrow$$

$||\Delta_n|| \to 0$. *Let* $\xi_k^n \in [x_{k-1}^n, x_k^n]$, *so that:*

$$\xi_k^n = \frac{k(n+1)}{n(n+1)} \Rightarrow \sigma_{\Delta_n}(f, \xi_k^n) = \sum_{k=1}^{n} f\left(\frac{k(k+1)}{n(n+1)}\right)\cdot\frac{2k}{n(n+1)} =$$

$$= \frac{2}{n(n+1)}\sum_{k=1}^{n} k\cdot\arctan\frac{k^2+k}{n^2+n} \Rightarrow$$

$$\Omega = \frac{1}{2}\lim_{n\to\infty}\frac{1}{n(n+1)}\sum_{k=1}^{n} k\arctan\frac{k^2+k}{n^2+n} = \frac{1}{2}\int_0^1 \arctan x\,dx$$

$$=\frac{1}{2}\int_0^1 x'\arctan x\,dx=\frac{1}{2}x\arctan x\Big|_0^1-\frac{1}{2}\int_0^1\frac{x}{1+x^2}dx$$

$$=\frac{\pi}{8}-\frac{1}{4}\ln(1+x^2)\Big|_0^1=\frac{\pi}{8}-\frac{1}{4}\ln 2$$