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$$\lim_{n\to\infty}\int_0^\infty \frac{n^2\sin\left(\frac{x}{n}\right)}{n^3x+x(1+x^3)}\,dx = ?$$

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Solution 1 by Ali Jaffal-Lebanon, Solution 2 by Naren Bhandari-Bajura-Nepal

Solution 1 by Ali Jaffal-Lebanon

$$\textit{Let } I_n = \int_0^{+\infty} \frac{n^2\sin\left(\frac{x}{n}\right)}{n^3x+x(1+x^3)}\,dx$$

$$= \int_0^1 \frac{n^2\sin\left(\frac{x}{n}\right)}{n^3x+x(1+x^3)}\,dx + \int_1^{+\infty} \frac{n^2\sin\left(\frac{x}{n}\right)}{n^3x+x(1+x^3)}\,dx$$

$$\textit{We have } \left|\frac{n^2\sin\left(\frac{x}{n}\right)}{n^3x+4x(1+x^3)}\right| \le \frac{nx}{n^3x+x(1+x^3)} \le \frac{n}{n^3+x^3+1}$$

$$\textit{Since } |\sin t| \le |t|, \forall t \in \mathbb{R}$$

$$\left|\int_0^1 \frac{n^2\sin\left(\frac{x}{n}\right)}{n^3x+x(1+x^3)}\,dx\right| \le \int_0^1 \frac{n}{n^3+x^3+1}\,dx \le \int_0^1 \frac{1}{n^2}\,dx \le \frac{1}{n^2}$$

$$\textit{then } \lim_{n\to\infty}\int_0^1 \frac{n^2\sin\left(\frac{x}{n}\right)}{n^3x+x(1+x^3)}\,dx = 0$$

we have:

$$\left|\int_1^{+\infty}\frac{n^2\sin\left(\frac{x}{n}\right)}{n^3x+x(1+x^3)}dx\right|\le\int_1^{+\infty}\frac{n}{n^3+x^3+1}dx\le\int_1^{+\infty}\frac{n}{n^3+x^2}dx\le\frac{1}{n^2}\int_1^{+\infty}\frac{dx}{1+\left(\frac{x}{n\sqrt{n}}\right)^2}$$

$$\le\frac{1}{\sqrt{n}}\left[\frac{\pi}{2}-\arctan\left(\frac{1}{n\sqrt{n}}\right)\right]=\frac{\pi}{2\sqrt{n}}$$

then $\lim_{n\to+\infty}\int_1^{+\infty}\frac{n^2\sin\left(\frac{x}{n}\right)}{n^3x+x(1+x^3)}dx=0$ ***therefore*** $\lim_{n\to+\infty}I_n=0$

Solution 2 by Naren Bhandari-Bajura-Nepal

Here $\lim_{n\to\infty}\int_0^\infty\frac{n^3\sin\left(\frac{x}{n}\right)}{n^3x+x(1+x^3)}dx$

Consider $f_n(x)=\frac{n^3\sin\left(\frac{x}{n}\right)}{n^3x+x(1+x^3)}$. ***We note that at*** $x=0$

$\lim_{n\to\infty}f_n(x)=0$ ***and*** $\forall x>0$ ***we have*** $\lim_{n\to\infty}f_n(1)=\lim_{n\to\infty}\frac{n^3\sin\frac{x}{n}}{n^3x+x(x^3+1)}=$

$$=\lim_{n\to\infty}\frac{n^3\sin\frac{x}{n}}{n^3x+x(1+x^3)}=\lim_{n\to\infty}\frac{n^3\left(\frac{x}{n}-0\left(\frac{x}{n}\right)^3\right)}{n^3x+n(1+x^3)}=\lim_{n\to\infty}\frac{n^2x}{n^3x+x(1+x^3)}=0$$

$|\lim_{n\to\infty}f_n(n)|\le g(n)=0$***. Then, by dominating convergence, we have:***

$$\lim_{n\to\infty}\int_0^n f_n(x)\,dx=\int_0^\infty g(n)=\int_0^\infty 0\,dx=0$$