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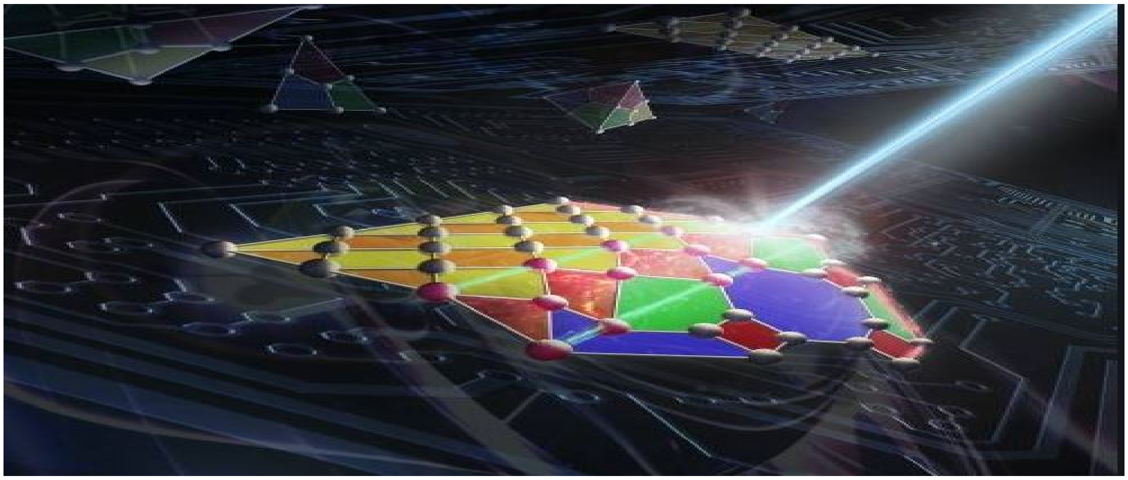

Dedicated to teacher Mehmet Șahin

$$f:\mathbb{R}\to\mathbb{R},\ 2f^4(x)+2f^2(x)+2\le 3f^3(x)+3f(x),\ \forall x\in\mathbb{R}$$

Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{n^2}\sum_{1\le i<j\le n}\left(f\left(e+\frac{\pi i}{n}\right)f\left(e+\frac{\pi j}{n}\right)\right)\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Florentin Vișescu – Romania

$$f:\mathbb{R}\to\mathbb{R},\ 2f^4(x)+2f^2(x)+2\le 3f^3(x)+3f(x),\ \forall x\in\mathbb{R}$$

$$2\left(f^4(x)+f^2(x)+1\right)\le 3\left[f^3(x)+f(x)\right]$$

$$2f^4(x)-3f^3(x)+2f^2(x)-3f(x)+2\le 0;\ \forall x\in\mathbb{R}$$

$f(x)=0$ ***not verify***

$$f(x)\ne 0,\ \forall x\in\mathbb{R}$$

$$2f^2(x)-3f(x)+2-3\frac{1}{f(x)}+\frac{2}{f^2(x)}\le 0;\ \forall x\in\mathbb{R}$$

$$2\left(f^2(x)+\frac{1}{f^2(x)}\right)-3\left(f(x)+\frac{1}{f(x)}\right)+2\le 0$$

$$f(x)+\frac{1}{f(x)}=g(x)$$

$$2\left(g^2(x)-2\right)-3g(x)+2\le 0$$

$$2g^2(x)-3g(x)-2\le 0$$

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$$\Delta = 9 + 16 = 25;\ g(x)_{1;2} = \frac{3 \pm 5}{4} = \left\langle \begin{matrix} 2 \\ -\frac{1}{2} \end{matrix} \right.$$

$$g(x) \in \left[-\frac{1}{2}; 2\right]$$

$$f(x) + \frac{1}{f(x)} \in \left[-\frac{1}{2}; 2\right];\ -\frac{1}{2} \le f(x) + \frac{1}{f(x)} \le 2$$

$$\Rightarrow f(x) = 1; \forall x \in \mathbb{R}$$

$$\Omega = \frac{1}{2}\left(\int_0^1 1\, dx\right)^2 = \frac{1}{2}\left(x\Big|_0^1\right)^2 = \frac{1}{2}$$

$$\sum_{1 \le i < j \le n}\left(f\left(e + \frac{\pi i}{n}\right) f\left(e + \frac{nj}{n}\right)\right) = \frac{\left(\sum_{i=1}^n f\left(e + \frac{\pi i}{n}\right)\right)^2 - \sum_{i=1}^n f^2\left(e + \frac{\pi i}{n}\right)}{2}$$

$$\lim_{n\to\infty} \frac{1}{n^2}\left(\sum_{1 \le i < j \le n}\left(f\left(e + \frac{\pi i}{n}\right) f\left(e + \frac{\pi j}{n}\right)\right)\right) =$$

$$= \frac{1}{2}\lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^n f\left(e + \frac{\pi i}{n}\right)\right]^2 - \frac{1}{2}\lim_{n\to\infty}\frac{1}{n}\cdot\frac{1}{n}\sum_{i=1}^n f^2\left(e + \frac{\pi i}{n}\right)$$

$$= \frac{1}{2}\left(\int_0^1 f(e + \pi x)\, dx\right)^2 - \frac{1}{2}\int_0^1 f^2(e + \pi x)\, dx \cdot \lim_{n\to\infty}\frac{1}{n} =$$

$$= \frac{1}{2}\left(\int_0^1 f(e + \pi x)\, dx\right)^2$$