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Find:

$$\Omega = \lim_{n\to\infty}\left(\log\left(\sum_{k=0}^{n}\left(2^{-k}\binom{n+k}{k}\right)\right)-\sum_{k=1}^{n}\frac{n}{n+k}\right)$$

Proposed by Florică Anastase – Romania

Solution 1 by Marian Ursărescu – Romania, Solution 2 by Mokhtar Khassani-Mostaganem-Algerie

Solution 1 by Marian Ursărescu – Romania

$$\text{Let } x_n = \sum_{k=0}^{n}\frac{C_{n+k}^{k}}{2^k}$$

$$\text{We have: } x_1 = \frac{C_1^0}{1}+\frac{C_2^1}{2} = 2$$

$$x_{n+1} = \sum_{k=0}^{n+1}\frac{C_{n+k+1}^{k}}{2^k} = \sum_{k=0}^{n}\frac{C_{n+k}^{k}+C_{n+k}^{k-1}}{2^k} =$$

$$= \sum_{k=0}^{n+1}\frac{C_{n+k}^{k}}{2^k}+\sum_{k=0}^{n+1}\frac{C_{n+k}^{k-1}}{2^k} = x_n+\sum_{k=1}^{n+2}\frac{C_{n+k}^{k-1}}{2^k} \quad (1)$$

$$\text{But } \sum_{k=1}^{n}\frac{C_{n+k}^{k-1}}{2^k} = \frac{C_{n+1}^{0}}{2}+\frac{C_{n+2}^{1}}{2^2}+\cdots+\frac{C_{2n+2}^{n+1}}{2^{n+2}} =$$

$$= \frac{1}{2}\left(\frac{C_{n+1}^{0}}{2^0}+\frac{C_{n+2}^{1}}{2}+\cdots+\frac{C_{2n+2}^{n+1}}{2^{n+2}}\right) = \frac{1}{2}x_{n+1} \quad (2)$$

$$\text{From (1)+(2)} \Rightarrow x_{n+1} = x_n+\frac{1}{2}x_{n+1} \Rightarrow \frac{1}{2}x_{n+1} = x_n \Rightarrow$$

$x_{n+1} = 2x_n$ *and* $x_1 = 2 \Rightarrow x_n = 2^n$ *(geometric progress)*

We must find:

$$\Omega = \lim_{n\to\infty}\left(\ln 2^n - \sum_{k=1}^{n}\frac{n}{n+k}\right) = \lim_{n\to\infty} n\left(\ln 2 - \sum_{k=1}^{n}\frac{1}{n+k}\right) =$$

$$= \lim_{n\to\infty}\frac{\ln 2 - \sum_{k=1}^{n}\frac{1}{n+k}}{\frac{1}{n}} \quad \textit{(3)}$$

Now, using Cesaro-Stolz for $\frac{0}{0}$***:***

$$a_n = \ln 2 - \sum_{k=1}^{n}\frac{1}{n+k}, b_n = \frac{1}{n}$$

a) b_n ***strictly decreasing***

b) $\lim_{n\to\infty} b_n = 0, \lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln 2 - \frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}} = \ln 2 - \int_0^1 \frac{1}{1+x}dx = \ln 2 -$

$\ln 2 = 0$

c) $\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty}\frac{-\frac{1}{2n+2}-\frac{1}{2n+1}+\frac{1}{n+1}}{\frac{1}{n+1}-\frac{1}{n}} =$

$$= \lim_{n\to\infty}\frac{-\frac{1}{2n+1}+\frac{1}{2n+2}}{\frac{n-n-1}{n(n+1)}} = \frac{\frac{-2n-2+2n+1}{(2n+1)(2n+2)}}{-\frac{1}{n(n+1)}} =$$

$$= \lim_{n\to\infty}\frac{n(n+1)}{(2n+1)(2n+2)} = \frac{1}{4} \quad \textit{(4)}$$

From (3)+(4)$\Rightarrow \Omega = \frac{1}{4}$

Solution 2 by Mokhtar Khassani-Mostaganem-Algerie

Let:

$$a_n = \sum_{k=0}^{n}\binom{n+k}{k}2^{-k} = \sum_{k=0}^{n}\binom{n+k-1}{k}2^{-k} + \sum_{k=0}^{n-1}\binom{n+k-1}{k}2^{-k-1} = 2^{-n}\binom{2n-1}{n} +$$

$$+\overbrace{\sum_{k=0}^{n-1}\binom{n+k-1}{k}2^{-k}}^{a_{n-1}} - \frac{1}{2}\left(2^{-n}\binom{2n}{n} - \overbrace{\sum_{k=0}^{n}\binom{n+k}{k}2^{-k}}^{a_n}\right)$$

$$\Rightarrow \alpha_n = 2\alpha_{n-1} + \binom{2n-1}{n}2^{1-n} - \binom{2n}{n}2^n =$$

$$= 2\alpha_{n-1} + \binom{2n-1}{n}2^{1-n} - \left(\binom{2n-1}{n} + \binom{2n-1}{n-1}\right)2^{-n} =$$

$$= 2\alpha_{n-1} + \binom{2n-1}{n}2^{-n} - \binom{2n-1}{n-1}2^{-n} = 2\alpha_{n-1}$$

$$\therefore \alpha_n = 2\alpha_{n-1} = 2^2\alpha_{n-1} = \cdots 2^n\alpha_0 = 2^n$$

Also:

$$\beta_n = \sum_{k=1}^{n}\frac{n}{n+k} = n\sum_{k=n+1}^{2n}\frac{1}{k} = n\left(\sum_{k=1}^{2n}\frac{1}{k} - \sum_{k=1}^{n}\frac{1}{k}\right) = n(H_{2n} - H_n)$$

Now:

$$\Omega = \lim_{n\to\infty}(\log(\alpha_n) - \beta_n) = \lim_{n\to\infty} n\,(H_n - H_{2n} + \log 2) =$$

$$= \lim_{n\to\infty}\frac{H_n - H_{2n} + \log 2}{\frac{1}{n}} \overset{S-C\left(\frac{0}{0}\right)}{=} \lim_{n\to\infty}\frac{H_{n+1} - H_{2n+2} + \log 2 - H_n + H_{2n} - \log 2}{\frac{1}{n+1} - \frac{1}{n}}$$

$$= \lim_{n\to\infty} n(n+1)\left(\frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1}\right) = \lim_{n\to\infty}\frac{n}{2(2n+1)} = \frac{1}{4}$$

$\Omega = \frac{1}{4}$ ***(answer)***