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Dacians, widely accepted as part of the Getae, were a branch of Thracians that inhabited Dacia (corresponding to Romania, Moldova and Northern Bulgaria).

The Dacian kingdom reached its maximum expansion during King Burebista, around 82 BC.



Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{\log_n^n \left(\frac{(1+H_1)^2 + (1+H_2)^2 + \dots + (1+H_n)^2}{n} \right)}{\log_n(1+H_1) \cdot \log_n(1+H_2) \cdot \dots \cdot \log_n(1+H_n)} \right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ali Jaffal-Lebanon, Solution 2 by Remus Florin Stanca-Romania

Solution 1 by Ali Jaffal-Lebanon

We have by GM-AM inequality:

$$\log^n \left(\frac{(1+H_1)^2 + (1+H_2)^2 + \dots + (1+H_n)^2}{n} \right) \geq$$

$$\log^n \left(\sqrt[n]{(1+H_1)^2(1+H_2)^2 \cdot \dots \cdot (1+H_n)^2} \right) \geq$$

$$\left(\frac{2}{n} \right)^n [\log(1+H_1)(1+H_2) \dots (1+H_n)]^n \geq$$

$$\left(\frac{2}{n} \right)^n [\log(1+H_1) + \log(1+H_2) + \dots + \log(1+H_n)]^n \quad (*)$$

$$\text{And } \log(1+H_1) + \dots + \log(1+H_n) \leq \left(\frac{1}{n} \right)^n [\log(1+H_1) + \dots + \log(1+H_n)]^n \quad (**)$$

$$\text{Let } U_n = \frac{\log_n^n((1+H_1)^2+(1+H_2)^2+\dots+(1+H_n)^2)}{\log_n(1+H_1) \cdot \log_n(1+H_2) \cdot \dots \cdot \log_n(1+H_n)}$$

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$$= \frac{\frac{1}{(\ln n)^n} \log^n \left(\frac{(1+H_1)^2 + \dots + (1+H_n)^2}{n} \right)}{\frac{1}{(\ln n)^n} [\log(1+H_1) \log(1+H_2) \dots \log(1+H_n)]} \stackrel{\text{by (*) and (**)}}{\geq} \left(\frac{2}{n} \right)^n \geq 2^n. \text{ So, } U_n \geq 2^n$$

but $\lim_{n \rightarrow +\infty} 2^n = +\infty$ then $\lim_{n \rightarrow \infty} U_n \geq +\infty$ so, $\lim_{n \rightarrow \infty} U_n = +\infty$

Solution 2 by Remus Florin Stanca-Romania

$$\begin{aligned} \Rightarrow \Omega &= \lim_{n \rightarrow \infty} \left(\frac{\left(\ln \left(\frac{(1+H_1)^2 + \dots + (1+H_n)^2}{n} \right) \right)^n}{(\ln(n))^n} \cdot \frac{(\ln(n))^n}{\ln(1+H_1) \cdot \dots \cdot \ln(1+H_n)} \right) = \\ &= \lim_{n \rightarrow \infty} \frac{\left(\ln \left(\frac{(1+H_1)^2 + \dots + (1+H_n)^2}{n} \right) \right)^n}{\ln(1+H_1) \cdot \dots \cdot \ln(1+H_n)}, \text{ let } x_n = \frac{\left(\ln \left(\frac{(1+H_1)^2 + \dots + (1+H_n)^2}{n} \right) \right)^n}{\ln(1+H_1) \cdot \dots \cdot \ln(1+H_n)} \Rightarrow \\ &\Rightarrow \frac{x_{n+1}}{x_n} = \left(\frac{\ln \left(\frac{(1+H_1)^2 + \dots + (1+H_{n+1})^2}{n+1} \right)}{\ln \left(\frac{(1+H_1)^2 + \dots + (1+H_n)^2}{n} \right)} \right)^n \cdot \frac{\ln \left(\frac{(1+H_1)^2 + \dots + (1+H_{n+1})^2}{n+1} \right)}{\ln(1+H_{n+1})} \quad (1) \\ &\lim_{n \rightarrow \infty} \frac{\ln \left(\frac{(1+H_1)^2 + \dots + (1+H_{n+1})^2}{n+1} \right)}{\ln(1+H_{n+1})} = \\ &= \lim_{n \rightarrow \infty} \frac{\ln \left(\frac{(1+H_1)^2 + \dots + (1+H_n)^2}{n} \right)}{\ln(1+H_n)} \stackrel{\text{Stolz-Cesaro}}{=} \\ &\stackrel{\text{Stolz-Cesaro}}{=} \lim_{n \rightarrow \infty} \frac{\ln \left(\frac{(1+H_1)^2 + \dots + (1+H_{n+1})^2}{(1+H_1)^2 + \dots + (1+H_n)^2} \cdot \frac{n}{n+1} \right)}{\ln \left(\frac{1+H_{n+1}}{1+H_n} \right)} = \\ &= \lim_{n \rightarrow \infty} \frac{\ln \left(\frac{(1+H_1)^2 + \dots + (1+H_{n+1})^2}{(1+H_1)^2 + \dots + (1+H_n)^2} \cdot \frac{n}{n+1} - 1 + 1 \right)}{\ln \left(\frac{1+H_{n+1}}{1+H_n} - 1 + 1 \right)} = \\ &= \lim_{n \rightarrow \infty} \frac{\ln \left(\frac{(1+H_1)^2 + \dots + (1+H_{n+1})^2}{(1+H_1)^2 + \dots + (1+H_n)^2} \cdot \frac{n}{n+1} - 1 + 1 \right)}{\frac{(1+H_1)^2 + \dots + (1+H_{n+1})^2}{(1+H_1)^2 + \dots + (1+H_n)^2} \cdot \frac{n}{n+1} - 1} \\ &= \frac{n((1+H_1)^2 + \dots + (1+H_{n+1})^2) - (n+1)((1+H_1)^2 + \dots + (1+H_n)^2)}{(n+1)((1+H_1)^2 + \dots + (1+H_n)^2)}. \end{aligned}$$

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$$\begin{aligned}
 & \frac{1}{(n+1)(1+H_n)} \cdot (n+1)(1+H_n) = \lim_{n \rightarrow \infty} (n+1) \frac{1+H_n}{\ln(n)} \cdot \ln(n) \cdot \\
 & \ln\left(\frac{1}{(n+1)(1+H_n)} + 1\right) \\
 & \frac{n((1+H_1)^2 + \dots + (1+H_{n+1})^2) - (n+1)((1+H_1)^2 + \dots + (1+H_n)^2)}{(n+1)((1+H_1)^2 + \dots + (1+H_n)^2)} = \\
 & = \lim_{n \rightarrow \infty} \ln(n) \cdot \frac{n(1+H_{n+1})^2 - (1+H_1)^2 - \dots - (1+H_n)^2}{(1+H_1)^2 + \dots + (1+H_n)^2} = \\
 & = \lim_{n \rightarrow \infty} \frac{n \ln^2(n)}{(1+H_1)^2 + \dots + (1+H_n)^2} \cdot \frac{1}{n \ln(n)} \cdot (n(1+H_{n+1})^2 - (1+H_1)^2 - \dots - (1+H_n)^2) = \\
 & \stackrel{\text{Stolz Cesaro}}{=} \lim_{n \rightarrow \infty} \left(\frac{n \ln^2(n) \left(\frac{n+1}{n} \left(\frac{\ln(n+1)^2}{\ln(n)} \right) - 1 \right)}{(1+H_{n+1})^2} \right) \cdot \\
 & \cdot \lim_{n \rightarrow \infty} \left(\frac{n(1+H_{n+1})^2 - (1+H_1)^2 - \dots - (1+H_n)^2}{n \ln(n)} \right) = \\
 & = \lim_{n \rightarrow \infty} \left(\left(\frac{\ln(n)}{1+H_{n+1}} \right)^2 \right) \cdot \lim_{n \rightarrow \infty} \left(n \ln \left(\frac{n+1}{n} \cdot \left(\frac{\ln(n+1)}{\ln(n)} \right)^2 \right) \right) \cdot \\
 & \cdot \lim_{n \rightarrow \infty} \left(\frac{n(1+H_{n+1})^2 - (1+H_1)^2 - \dots - (1+H_n)^2}{n \ln(n)} \right) = \\
 & = 1 \cdot \ln(e) \cdot \lim_{n \rightarrow \infty} \left(\frac{n(1+H_{n+1})^2 - (1+H_1)^2 - \dots - (1+H_n)^2}{n \ln(n)} \right) \stackrel{\text{Stolz Cesaro}}{=} \\
 & \stackrel{\text{Stolz Cesaro}}{=} \lim_{n \rightarrow \infty} \frac{(n+1)(1+H_{n+2})^2 - n(1+H_{n+1})^2 - (1+H_{n+1})^2}{(n+1) \ln(n+1) - n \ln(n)} = \\
 & = \lim_{n \rightarrow \infty} (n+1) \cdot \frac{(2+2H_{n+1}+\frac{1}{n+2}) \cdot \frac{1}{n+1}}{n \ln(n) \ln\left(\frac{n+1}{n} \cdot \frac{\ln(n+1)}{\ln(n)}\right)} = \lim_{n \rightarrow \infty} \frac{2+2H_{n+1}+\frac{1}{n+2}}{\ln(n)} \stackrel{\text{Stolz Cesaro}}{=} 2 \quad (2) \\
 & \lim_{n \rightarrow \infty} e^{\frac{\ln\left(\frac{(1+H_1)^2 + \dots + (1+H_{n+1})^2}{(1+H_1)^2 + \dots + (1+H_n)^2} \cdot \frac{n}{n+1}\right)}{\ln\left(\frac{(1+H_1)^2 + \dots + (1+H_n)^2}{n}\right)} \cdot n} = \\
 & = \lim_{n \rightarrow \infty} e^{n \ln\left(\frac{(1+H_1)^2 + \dots + (1+H_{n+1})^2}{(1+H_1)^2 + \dots + (1+H_n)^2} \cdot \frac{n}{n+1}\right) \cdot \frac{\ln(1+H_n)}{\ln\left(\frac{(1+H_1)^2 + \dots + (1+H_n)^2}{n}\right)} \cdot \frac{1}{\ln(1+H_n)}} =
 \end{aligned}$$

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$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} e^{\frac{1}{2} \cdot n \cdot \frac{1}{\ln(1+H_n)} \cdot \frac{\ln\left(\frac{(1+H_1)^2 + \dots + (1+H_{n+1})^2}{n+1} \cdot \frac{n}{n+1}\right)}{\ln\left(\frac{1+H_{n+1}}{1+H_n}\right)}} = \\
 &= \lim_{n \rightarrow \infty} e^{\frac{n}{\ln(1+H_n)} \cdot \ln\left(\frac{1+H_{n+1}}{1+H_n}\right)} = \lim_{n \rightarrow \infty} e^{\frac{n}{\ln(1+H_n)} \cdot \frac{\ln\left(\frac{1}{(n+1)(1+H_n)} + 1\right)}{\frac{1}{(n+1)(1+H_n)}(n+1)(1+H_n)}} = \\
 &= e^{\frac{1}{\infty}} = e^0 = 1 \quad (3) \quad \stackrel{(1):(2):(3)}{\Rightarrow} \quad \lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = 2 > 1 \Rightarrow \lim_{n \rightarrow \infty} x_n = \infty \Rightarrow \Omega = \infty
 \end{aligned}$$

$$\text{Because } \lim_{n \rightarrow \infty} \left(\frac{\ln\left(\frac{(1+H_1)^2 + \dots + (1+H_{n+1})^2}{n+1}\right)}{\ln\left(\frac{(1+H_1)^2 + \dots + (1+H_n)^2}{n}\right)} \right)^n = \lim_{n \rightarrow \infty} e^{\frac{n \cdot \frac{\ln\left(\frac{(1+H_1)^2 + \dots + (1+H_{n+1})^2}{n+1} \cdot \frac{n}{n+1}\right)}{\ln\left(\frac{(1+H_1)^2 + \dots + (1+H_n)^2}{n}\right)}}}{\ln\left(\frac{(1+H_1)^2 + \dots + (1+H_n)^2}{n}\right)}}$$