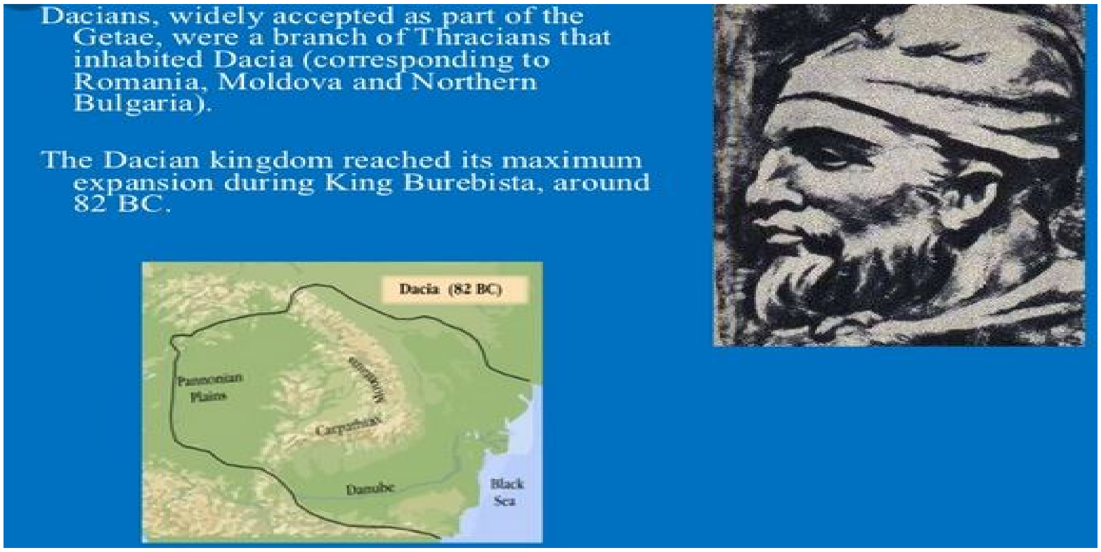
R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro



Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{\log_n^n\left(\frac{(1+H_1)^2+(1+H_2)^n+\cdots+(1+H_n)^2}{n}\right)}{\log_n(1+H_1)\cdot\log_n(1+H_2)\cdot\ldots\cdot\log_n(1+H_n)}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ali Jaffal-Lebanon, Solution 2 by Remus Florin Stanca-Romania

Solution 1 by Ali Jaffal-Lebanon

We have by GM-AM inequality:

$$\log^n\left(\frac{(1+H_1)^2+(1+H_2)^2+\cdots+(1+H_n)^2}{n}\right)\geq$$

$$\log^n\left(\sqrt[n]{(1+H_1)^2(1+H_2)^2\cdot\ldots\cdot(1+H_n)^2}\right)\geq$$

$$\left(\frac{2}{n}\right)^n[\log(1+H_1)(1+H_2)\ldots(1+H_n)]^n\geq$$

$$\left(\frac{2}{n}\right)^n[\log(1+H_1)+\log(1+H_2)+\cdots+\log(1+H_n)]^n \quad (*)$$

And $\log(1+H_1)+\cdots+\log(1+H_n)\leq\left(\frac{1}{n}\right)^n[\log(1+H_1)+\cdots+\log(1+H_n)]^n$ ***(**)***

Let $U_n=\frac{\log_n^n((1+H_1)^2+(1+H_2)^2+\cdots+(1+H_n)^2)}{\log_n(1+H_1)\cdot\log_n(1+H_2)\cdot\ldots\cdot\log_n(1+H_n)}$

$$=\frac{\frac{1}{(\ln n)^n}\log^n\left(\frac{(1+H_1)^2+\cdots+(1+H_n)^2}{n}\right)}{\frac{1}{(\ln n)^n}[\log(1+H_1)\log(1+H_2)\cdot\ldots\cdot\log(1+H_n)]}\overset{by\ (*)\ and\ (**)}{\geq}\frac{\left(\frac{2}{n}\right)^n}{\left(\frac{1}{n}\right)^n}\geq 2^n.\ \text{So, } U_n\geq 2^n$$

but $\lim_{n\to+\infty}2^n=+\infty$ ***then*** $\lim_{n\to\infty}U_n\geq+\infty$ **so,** $\lim_{n\to\infty}U_n=+\infty$

Solution 2 by Remus Florin Stanca-Romania

$$\Rightarrow\Omega=\lim_{n\to\infty}\left(\frac{\left(\ln\left(\frac{(1+H_1)^2+\cdots+(1+H_n)^2}{n}\right)\right)^n}{(\ln(n))^n}\cdot\frac{(\ln(n))^n}{\ln(1+H_1)\cdot\ldots\cdot\ln(1+H_n)}\right)=$$

$$=\lim_{n\to\infty}\frac{\left(\ln\left(\frac{(1+H_1)^2+\cdots+(1+H_n)^2}{n}\right)\right)^n}{\ln(1+H_1)\cdot\ldots\cdot(1+H_n)},\ \text{let } x_n=\frac{\left(\ln\left(\frac{(1+H_1)^2+\cdots+(1+H_n)^2}{n}\right)\right)^n}{\ln(1+H_1)\cdot\ldots\cdot\ln(1+H_n)}\Rightarrow$$

$$\Rightarrow\frac{x_{n+1}}{x_n}=\left(\frac{\ln\left(\frac{(1+H_1)^2+\cdots+(1+H_{n+1})^2}{n+1}\right)}{\ln\left(\frac{(1+H_1)^2+\cdots+(1+H_n)^2}{n}\right)}\right)^n\cdot\frac{\ln\left(\frac{(1+H_1)^2+\cdots+(1+H_{n+1})^2}{n+1}\right)}{\ln(1+H_{n+1})}\quad(1)$$

$$\lim_{n\to\infty}\frac{\ln\left(\frac{(1+H_1)^2+\cdots+(1+H_{n+1})^2}{n+1}\right)}{\ln(1+H_{n+1})}=$$

$$=\lim_{n\to\infty}\frac{\ln\left(\frac{(1+H_1)^2+\cdots+(1+H_n)^2}{n}\right)}{\ln(1+H_n)}\overset{Stolz-Cesaro}{=}$$

$$\overset{Stolz-Cesaro}{=}\lim_{n\to\infty}\frac{\ln\left(\frac{(1+H_1)^2+\cdots+(1+H_{n+1})^2}{(1+H_1)^2+\cdots+(1+H_n)^2}\cdot\frac{n}{n+1}\right)}{\ln\left(\frac{1+H_{n+1}}{1+H_n}\right)}=$$

$$=\lim_{n\to\infty}\frac{\ln\left(\frac{(1+H_1)^2+\cdots+(1+H_{n+1})^2}{(1+H_1)^2+\cdots+(1+H_n)^2}\cdot\frac{n}{n+1}-1+1\right)}{\ln\left(\frac{1+H_{n+1}}{1+H_n}-1+1\right)}=$$

$$=\lim_{n\to\infty}\frac{\ln\left(\frac{(1+H_1)^2+\cdots+(1+H_{n+1})^2}{(1+H_1)^2+\cdots+(1+H_n)^2}\cdot\frac{n}{n+1}-1+1\right)}{\frac{(1+H_1)^2+\cdots+(1+H_{n+1})^2}{(1+H_1)^2+\cdots+(1+H_n)^2}\cdot\frac{n}{n+1}-1}\cdot$$

$$\cdot\frac{n((1+H_1)^2+\cdots+(1+H_{n+1})^2)-(n+1)((1+H_1)^2+\cdots(1+H_n)^2)}{(n+1)((1+H_1)^2+\cdots+(1+H_n)^2)}\cdot$$

$$\cdot \frac{\frac{1}{(n+1)(1+H_n)}}{\ln\left(\frac{1}{(n+1)(1+H_n)}+1\right)} \cdot (n+1)(1+H_n) = \lim_{n\to\infty}(n+1)\frac{1+H_n}{\ln(n)} \cdot \ln(n) \cdot$$

$$\cdot \frac{n((1+H_1)^2+\cdots+(1+H_{n+1})^2)-(n+1)((1+H_1)^2+\cdots(1+H_n)^2)}{(n+1)((1+H_1)^2+\cdots+(1+H_n)^2)} =$$

$$= \lim_{n\to\infty} \ln(n) \cdot \frac{n(1+H_{n+1})^2-(1+H_1)^2-\cdots-(1+H_n)^2}{(1+H_1)^2+\cdots+(1+H_n)^2} =$$

$$= \lim_{n\to\infty} \frac{n\ln^2(n)}{(1+H_1)^2+\cdots+(1+H_n)^2} \cdot \frac{1}{n\ln(n)} \cdot \left(n(1+H_{n+1})^2-(1+H_1)^2-\cdots-(1+H_n)^2\right) =$$

$$\overset{Stolz\ Cesaro}{=} \lim_{n\to\infty}\left(\frac{n\ln^2(n)\left(\frac{n+1}{n}\left(\frac{\ln(n+1)^2}{\ln(n)}\right)-1\right)}{(1+H_{n+1})^2}\right) \cdot$$

$$\cdot \lim_{n\to\infty}\left(\frac{n(1+H_{n+1})^2-(1+H_1)^2-\cdots-(1+H_n)^2}{n\ln(n)}\right) =$$

$$= \lim_{n\to\infty}\left(\left(\frac{\ln(n)}{1+H_{n+1}}\right)^2\right) \cdot \lim_{n\to\infty}\left(n\ln\left(\frac{n+1}{n}\cdot\left(\frac{\ln(n+1)}{\ln(n)}\right)^2\right)\right) \cdot$$

$$\cdot \lim_{n\to\infty}\left(\frac{n(1+H_{n+1})^2-(1+H_1)^2-\cdots-(1+H_n)^2}{n\ln(n)}\right) =$$

$$= 1 \cdot \ln(e) \cdot \lim_{n\to\infty}\left(\frac{n(1+H_{n+1})^2-(1+H_1)^2-\cdots-(1+H_n)^2}{n\ln(n)}\right) \overset{Stolz\ Cesaro}{=}$$

$$\overset{Stolz\ Cesaro}{=} \lim_{n\to\infty} \frac{(n+1)(1+H_{n+2})^2-n(1+H_{n+1})^2-(1+H_{n+1})^2}{(n+1)\ln(n+1)-n\ln(n)} =$$

$$= \lim_{n\to\infty}(n+1) \cdot \frac{\left(2+2H_{n+1}+\frac{1}{n+2}\right)\cdot\frac{1}{n+1}}{n\ln(n)\ln\left(\frac{n+1}{n}\cdot\frac{\ln(n+1)}{\ln(n)}\right)} = \lim_{n\to\infty}\frac{2+2H_{n+1}+\frac{1}{n+2}}{\ln(n)} \overset{Stolz\ Cesaro}{=} 2 \quad \textbf{(2)}$$

$$\lim_{n\to\infty} e^{\frac{\ln\left(\frac{(1+H_1)^2+\cdots+(1+H_{n+1})^2}{(1+H_1)^2+\cdots+(1+H_n)^2}\cdot\frac{n}{n+1}\right)}{\ln\left(\frac{(1+H_1)^2+\cdots+(1+H_n)^2}{n}\right)}\cdot n} =$$

$$= \lim_{n\to\infty} e^{n\ln\left(\frac{(1+H_1)^2+\cdots+(1+H_{n+1})^2}{(1+H_1)^2+\cdots+(1+H_n)^2}\cdot\frac{n}{n+1}\right)\cdot\frac{\ln(1+H_n)}{\ln\left(\frac{(1+H_1)^2+\cdots+(1+H_n)^2}{n}\right)}\cdot\frac{1}{\ln(1+H_n)}} =$$

$$=\lim_{n\to\infty} e^{\frac{1}{2}\cdot n\cdot\frac{1}{\ln(1+H_n)}\cdot\frac{\ln\left(\frac{(1+H_1)^2+\cdots+(1+H_{n+1})^2}{(1+H_1)^2+\cdots+(1+H_n)^2}\cdot\frac{n}{n+1}\right)}{\ln\left(\frac{1+H_{n+1}}{1+H_n}\right)}\ln\left(\frac{1+H_{n+1}}{1+H_n}\right)} =$$

$$=\lim_{n\to\infty} e^{\frac{n}{\ln(1+H_n)}\cdot\ln\left(\frac{1+H_{n+1}}{1+H_n}\right)} = \lim_{n\to\infty} e^{\frac{n}{\ln(1+H_n)}\cdot\frac{\ln\left(\frac{1}{(n+1)(1+H_n)}+1\right)}{\frac{1}{(n+1)(1+H_n)}(n+1)(1+H_n)}} =$$

$$= e^{\frac{1}{\infty}} = e^0 = 1 \quad \textit{(3)} \overset{(1);(2);(3)}{\Rightarrow} \lim_{n\to\infty}\frac{x_{n+1}}{x_n} = 2 > 1 \Rightarrow \lim_{n\to\infty} x_n = \infty \Rightarrow \Omega = \infty$$

Because $\lim_{n\to\infty}\left(\frac{\ln\left(\frac{(1+H_1)^2+\cdots+(1+H_{n+1})^2}{n+1}\right)}{\ln\left(\frac{(1+H_1)^2+\cdots+(1+H_n)^2}{n}\right)}\right)^n = \lim_{n\to\infty} e^{n\cdot\frac{\ln\left(\frac{(1+H_1)^2+\cdots+(1+H_{n+1})^2}{(1+H_1)^2+\cdots+(1+H_n)^2}\cdot\frac{n}{n+1}\right)}{\ln\left(\frac{(1+H_1)^2+\cdots+(1+H_n)^2}{n}\right)}}$