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If $x, y, z > 0$, different in pairs, then:

$$3+\Omega(x,y)+\Omega(y,z)+\Omega(z,x)>\log\left(\prod_{cyc}\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)\right)$$

$$\Omega(x,y)=\sum_{k=1}^{\infty}\left(\frac{1}{2k}\left(\frac{x^2-2xy+y^2}{x^2+2xy+y^2}\right)^k\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India, Solution 2 by Adrian Popa-Romania, Solution 3 by Remus Florin Stanca-Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\Omega(x,y)=\sum_{k=1}^{\infty}\frac{1}{2k}\left(\frac{x^2-2xy+y^2}{x^2+2xy+y^2}\right)^k=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{x^2-2xy+y^2}{x^2+2xy+y^2}\right)^k$$

$$=-\frac{1}{2}\ln\left(1-\frac{x^2-2xy+y^2}{x^2+2xy+y^2}\right)=-\frac{1}{2}\ln\left(\frac{4xy}{x^2+2xy+y^2}\right)=$$

$$=\ln\left(\frac{x+y}{2\sqrt{xy}}\right)=\ln\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)-\ln 2$$

$$\therefore 3\ln e+\sum_{cyc}\Omega(x,y)=\sum_{cyc}\ln\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)+3\ln\left(\frac{e}{2}\right)=$$

$$=\sum_{cyc}\ln\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)+3\ln(1.3591)$$

$$3+\sum_{cyc}\Omega(x,y)>\sum_{cyc}\ln\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)$$

(Proved)

Solution 2 by Adrian Popa-Romania

$$\therefore\ \left.\begin{array}{c}1+x+x^2+\cdots+x^{n-1}=\dfrac{x^n-1}{x-1}\\ If\ x\in(0,1)and\ n\to\infty\Rightarrow x^n\to 0\end{array}\right\}\Rightarrow$$

$$\Rightarrow 1+x+x^2+\cdots+x^{n-1}+\cdots=\frac{1}{1-x}\Big|\int\ \Rightarrow$$

$$\Rightarrow x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{x^n}{n}=-\ln(1-x)\Rightarrow\sum_{k=1}^{\infty}\frac{x^k}{k}=-\ln(1-x)$$

$$\Omega(x,y)=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}\left(\left(\frac{x-y}{x+y}\right)^2\right)^k=-\frac{1}{2}\cdot\ln\left(1-\frac{(x-y)^2}{(x+y)^2}\right)=-\frac{1}{2}\ln\frac{4xy}{(x+y)^2}=$$

$$=\frac{1}{2}\ln\frac{(x+y)^2}{4xy}=\frac{1}{2}\ln\left(\frac{x+y}{2\sqrt{xy}}\right)^2=\ln\frac{x+y}{2\sqrt{x\cdot y}}=\ln\left(\frac{\sqrt{x}}{2\sqrt{y}}+\frac{\sqrt{y}}{2\sqrt{x}}\right)$$

We must prove that: $3+\sum\ln\left(\frac{\sqrt{x}}{2\sqrt{y}}+\frac{\sqrt{y}}{2\sqrt{x}}\right)\geq\sum\ln\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)$

$$\underbrace{1}_{\ln e}+\ln\left(\frac{1}{2}\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)\right)=\ln\left(\frac{e}{2}\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)\right)\overset{?}{>}\ln\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)\Leftrightarrow$$

$\Leftrightarrow\ln\frac{e}{2}+\ln\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)>\ln\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)\Rightarrow\ln\frac{e}{2}\geq 1$ ***(True) because*** $e>2$ ***(1)***

Similarly: $1+\ln\left(\frac{1}{2}\left(\sqrt{\frac{x}{z}}+\sqrt{\frac{z}{x}}\right)\right)>\ln\left(\sqrt{\frac{x}{z}}+\sqrt{\frac{z}{x}}\right)$ ***(2) and***

$$1+\ln\left(\frac{1}{2}\left(\sqrt{\frac{y}{z}}+\sqrt{\frac{z}{y}}\right)\right)>\ln\left(\sqrt{\frac{y}{z}}+\sqrt{\frac{z}{y}}\right)\quad (3)$$

$$(1)+(2)+(3)\Rightarrow 3+\Omega(x,y)+\Omega(y,z)+\Omega(x,z)\geq\ln\left(\prod\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)\right)$$

Solution 3 by Remus Florin Stanca-Romania

$\Omega(x,y)=\sum_{k=1}^{\infty}\left(\frac{1}{2k}\left(\frac{x-y}{x+y}\right)^{2k}\right)$, ***let*** $\frac{x-y}{x+y}=\alpha$, ***we also know that*** $x,y,z>0\Rightarrow$

$\Rightarrow -y<y\Rightarrow x-y<x+y\Rightarrow\frac{x-y}{x+y}<1$ ***because*** $x+y>0$, ***so*** $\alpha<1$ ***(1)***

$$\Omega(x,y)=\sum_{k=1}^{\infty}\left(\frac{\alpha^{2k}}{2k}\right)=\sum_{k=1}^{\infty}\left(\int\left(\frac{\alpha^{2k}}{2k}\right)'d\alpha\right)=$$

$$=\sum_{k=1}^{\infty}\left(\int\alpha^{2k-1}\,d\alpha\right)=\int\left(\sum_{k=1}^{\infty}\alpha^{2k-1}\right)d\alpha=\int\lim_{n\to\infty}\left(\alpha\cdot\frac{(\alpha^2)^n-1}{\alpha^2-1}\right)d\alpha=-\int\frac{\alpha}{\alpha^2-1}d\alpha=$$

$$=-\frac{1}{2}\int\frac{2\alpha}{\alpha^2-1}d\alpha=-\frac{1}{2}\ln(|\alpha^2-1|)\overset{\alpha\leq 1}{=}-\frac{1}{2}\ln(1-\alpha^2)$$

$$=-\frac{1}{2}\ln\left(1-\frac{x^2-2xy+y^2}{x^2+2xy+y^2}\right)=$$

$=-\frac{1}{2}\ln\left(\frac{4xy}{(x+y)^2}\right)=-\ln\left(\frac{2\sqrt{xy}}{x+y}\right)$ ***because*** $x,y>0$

$$\Rightarrow\Omega(x,y)=-\ln\left(\frac{2\sqrt{xy}}{x+y}\right)\Rightarrow 3+\Omega(x,y)+\Omega(y,z)+\Omega(z,x)=$$

$$=3-\sum_{cyc}\ln\left(\frac{2\sqrt{xy}}{x+y}\right)=3+\ln\left(\frac{\prod_{cyc}\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)}{8}\right)=$$

$$=3-\ln(8)+\ln\left(\prod_{cyc}\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)\right)\quad (2)$$

we know that $2<e\Rightarrow 8<e^3\Rightarrow\ln(8)<3\Rightarrow 3-\ln(8)>0\Rightarrow$

$$\Rightarrow 3+\ln\left(\prod_{cyc}\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)\right)-\ln(8)>\ln\left(\prod_{cyc}\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)\right)\overset{(2)}{\Rightarrow}$$

$$\Rightarrow 3+\Omega(x,y)+\Omega(y,z)+\Omega(z,x)>\ln\left(\prod_{cyc}\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)\right)$$

(Q.E.D.)