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If $a,b,c>0, n\in\mathbb{N}, n\geq 2$ then:

$$\int_a^{2a}\int_b^{2b}\int_c^{2c}\left(\sqrt[n]{\frac{x+1}{y+1}}+\sqrt[n+2]{\frac{y+1}{z+1}}+\sqrt[n+4]{\frac{z+1}{x+1}}\right)dxdydz>15abc$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution by Daniel Sitaru-Romania

$$\sqrt[n]{\frac{x+1}{y+1}}+\sqrt[n+2]{\frac{y+1}{z+1}}+\sqrt[n+4]{\frac{z+1}{x+1}}=n\cdot\frac{\sqrt[n]{\frac{x+1}{y+1}}}{n}+(n+2)\cdot\frac{\sqrt[n+2]{\frac{y+1}{z+1}}}{n+2}+(n+4)\cdot\frac{\sqrt[n+4]{\frac{z+1}{x+1}}}{n+4}\geq$$

$$\overset{AM-GM}{\geq}(3n+6)\sqrt[3n+6]{\left(\frac{\sqrt[n]{\frac{x+1}{y+1}}}{n}\right)^n\cdot\left(\frac{\sqrt[n+2]{\frac{y+1}{z+1}}}{n+2}\right)^{n+2}\cdot\left(\frac{\sqrt[n+4]{\frac{z+1}{x+1}}}{n+4}\right)^{n+4}}=$$

$$=(3n+6)\sqrt[3n+6]{\frac{\frac{x+1}{y+1}\cdot\frac{y+1}{z+1}\cdot\frac{z+1}{x+1}}{n^n\cdot(n+2)^{n+2}\cdot(n+4)^{n+4}}}=(3n+6)\sqrt[3n+6]{\frac{1}{n^n\cdot(n+2)^{n+2}\cdot(n+4)^{n+4}}}\geq$$

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$$\overset{AM-GM}{\geq} \frac{3n+6}{\frac{n\cdot n+(n+2)\cdot(n+2)+(n+4)\cdot(n+4)}{n+n+2+n+4}} = \frac{9(n^2+4n+4)}{3n^2+12n+20} > \frac{5}{2} \leftrightarrow$$

$$\leftrightarrow 18n^2+72n+72 > 15n^2+60n+100 \leftrightarrow 3n^2+12n > 28 \ (n \geq 2)$$

$$\sqrt[n]{\frac{x+1}{y+1}} + \sqrt[n+2]{\frac{y+1}{z+1}} + \sqrt[n+4]{\frac{z+1}{x+1}} > \frac{5}{2} \rightarrow \int_a^{2a}\int_b^{3b}\int_c^{4c}\left(\sqrt[n]{\frac{x+1}{y+1}} + \sqrt[n+2]{\frac{y+1}{z+1}} + \sqrt[n+4]{\frac{z+1}{x+1}}\right)dxdydz >$$

$$> \int_a^{2a}\int_b^{3b}\int_c^{4c}\frac{5}{2}dxdydz = \frac{5}{2}(2a-a)(3b-b)(4c-c) = 15abc$$