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ROMANIAN MATHEMATICAL MAGAZINE

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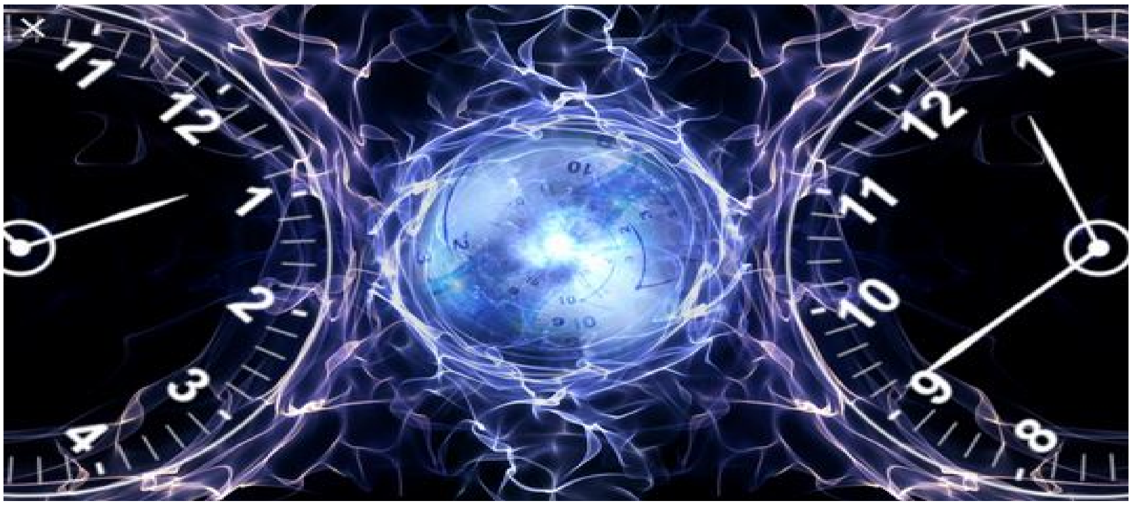


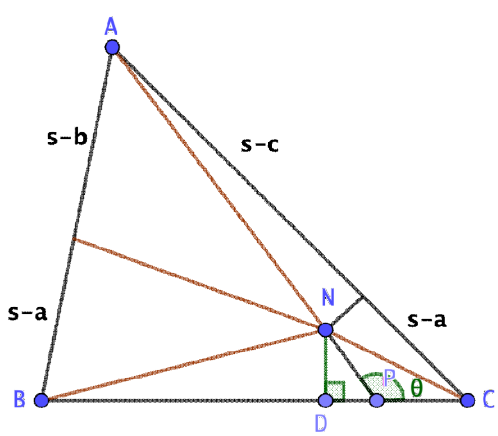
In $\Delta ABC, N$ – Nagel's point, $ND \perp BC, NE \perp AC, NF \perp AB, D \in (BC)$, $E \in (CA), F \in (AB)$. Prove that:

$$\frac{r_a}{ND} + \frac{r_b}{NE} + \frac{r_c}{NF} \geq \left(\frac{3R}{2r}\right)^2 \geq 9$$

Proposed by Marian Ursărescu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{r_a}{ND} + \frac{r_b}{NE} + \frac{r_c}{NF} \overset{(i)}{\geq} \left(\frac{3R}{2r}\right)^2 \geq 9$$



Van Aubel's theorem $\Rightarrow \frac{AN}{n_a - AN} = \frac{s-c}{s-a} + \frac{s-b}{s-a} = \frac{a}{s-a}$

$$\Rightarrow \frac{n_a - AN}{AN} = \frac{s-a}{a} \Rightarrow \frac{n_a}{AN} = \frac{s-a+a}{a} = \frac{s}{a} \Rightarrow \frac{AN}{n_a} \overset{(1)}{=} \frac{a}{s}$$

Sine – rule on $\Delta APC \Rightarrow \frac{b}{\sin\theta} = \frac{n_a}{\sin C} \Rightarrow \sin(180^\circ - \theta) = \frac{bc}{2Rn_a}$

$\Rightarrow \frac{ND}{NP} = \frac{bc}{2Rn_a}$ ***(using*** ΔNDP***)***

$$\Rightarrow \frac{ND}{n_a - AN} = \frac{bc}{2Rn_a} \Rightarrow \frac{n_a - AN}{n_a} = \frac{2R\cdot ND\cdot a}{abc} \Rightarrow 1 - \frac{AN}{n_a} = \frac{2R\cdot ND\cdot a}{4Rrs}$$

$$\overset{by\ (1)}{\Rightarrow} 1 - \frac{a}{s} = \frac{a\cdot ND}{2rs} \Rightarrow \frac{s-a}{s} = \frac{a\cdot ND}{2rs} \Rightarrow ND \overset{(a)}{=} 2r\left(\frac{s-a}{a}\right)$$

Similarly, $NE \overset{(b)}{=} 2r\left(\frac{s-b}{b}\right)$ ***and*** $NF \overset{(c)}{=} 2r\left(\frac{s-c}{c}\right)$

(a), (b), (c) $\Rightarrow$ ***LHS of (i)*** $= \sum \frac{r_a a}{2r(s-a)} = \frac{1}{2r}\sum \frac{r_a(a-s+s)}{s-a} = \frac{1}{2r}\sum\left(-r_a + \frac{r_a}{r}\cdot\frac{rs}{s-a}\right)$

$$= \frac{1}{2r}\sum\left(-r_a + \frac{1}{r}r_a^2\right) = \frac{(4R+r)^2 - 2s^2}{2r^2} - \frac{4R+r}{2r}\left(\because \sum r_a^2 = (4R+r)^2 - 2s^2\right)$$

$$= \frac{(4R+r)^2 - 2s^2 - r(4R+r)}{2r^2} = \frac{8R^2 + 2Rr - s^2}{r^2} \geq \frac{9R^2}{4r^2}$$

$$\Leftrightarrow 32R^2 + 8Rr - 4s^2 \geq 9R^2 \Leftrightarrow 4s^2 \overset{(ii)}{\leq} 23R^2 + 8Rr$$

Now, LHS of (ii) $\overset{Gerretsen}{\leq} 16R^2 + 16Rr + 12r^2 \overset{?}{\leq} 23R^2 + 8Rr$

$$\Leftrightarrow 7R^2 - 8Rr - 12r^2 \overset{?}{\geq} 0 \Leftrightarrow (R-2r)(7R+6r) \overset{?}{\geq} 0$$

$\rightarrow$ ***true*** $\Rightarrow$ ***(ii)*** $\Rightarrow$ ***(i) is true and*** $\because R \overset{Euler}{\geq} 2r \therefore \left(\frac{3R}{2r}\right)^2 \geq 9$ ***and thus the proposed chain of inequalities is true (Proved)***