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In ΔABC the following relationship holds:

$$\sum_{cyc}\left(w_a\sqrt{\frac{r_a}{h_a}}\right)\le\frac{3R}{2}\sqrt{1+\frac{8m_am_bm_c}{h_ah_bh_c}}$$

Proposed by Mokhtar Khassani-Mostaganem-Algerie

Solution 1 by Marian Ursărescu-Romania, Solution 2 by Soumava Chakraborty-Kolkata-India

Solution 1 by Marian Ursărescu-Romania

In any Δ we have $w_a\le\sqrt{s(s-a)};\ s=\frac{a+b+c}{2}$

$$w_a\sqrt{\frac{r_a}{h_a}}\le\sqrt{s(s-a)}\sqrt{\frac{\frac{S}{s-a}}{\frac{2S}{a}}}=\sqrt{\frac{as}{2}}\Rightarrow \text{we must show:}\sum\sqrt{\frac{as}{2}}\le\frac{3R}{2}\sqrt{1+\frac{8m_am_bm_c}{h_ah_bh_c}}\Leftrightarrow$$

$$\frac{s}{2}\left(\sum\sqrt{a}\right)^2\le\frac{9R^2}{4}\left(1+\frac{8m_am_bm_c}{h_ah_bh_c}\right)\quad (1)$$

From Cauchy's inequality: $\left(\sum\sqrt{a}\right)^2\le 3\sum a=6s\quad$ *(2)*

From (1)+(2) we must show: $3s^2\le\frac{9R^2}{4}\left(1+\frac{8m_am_bm_c}{h_ah_bh_c}\right)\quad$ *(3)*

From Mitrinovic's inequality: $s^2\le\frac{27}{4}R^2\quad$ *(4)*

From (3)+(4) we must show: $9\le 1+\frac{8m_am_bm_c}{h_ah_bh_c}\Leftrightarrow 8\le\frac{8m_am_bm_c}{h_ah_bh_c}\Leftrightarrow$

$h_ah_bh_c\le m_am_bm_c$*, obviously it is true.*

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS=\sum\frac{2bc\cos\frac{A}{2}}{b+c}\sqrt{\frac{sa}{2bc}\cdot\frac{bc}{s(s-a)}}=\sum\left(\frac{2\sqrt{bc}}{b+c}\sqrt{\frac{as}{2}}\right)\overset{A-G}{\leq}\sum\sqrt{\frac{as}{2}}\overset{CBS}{\leq}\sqrt{\frac{3s\cdot 2s}{2}}=\sqrt{3}s\overset{Mitrinovic}{\leq}$$

$$\leq\sqrt{3}\left(\frac{3\sqrt{3}R}{2}\right)=\frac{9R}{2}\leq\frac{3R}{2}\sqrt{1+8\pi\frac{m_a}{h_a}}\ (\because m_a\geq h_a, etc)$$ ***(Done)***