PLAGIOGONAL PLANE COORDINATE SYSTEM (of same unit lengths) - POLYGON AREA (n-gon)

authors : Thanasis Gakopoulos

Dimitris Blatsis

A) Quadrilateral area

It has been proved and is known that the area of a triangle ABC in plagiogonal system is given by the formula:

$$(\mathrm{ABC}) = \frac{\sin\theta}{2}\left\| \begin{matrix} 1 & 1 & 1 \\ \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \end{matrix} \right\| = \frac{\sin\theta}{2}\left|\det\left(\overrightarrow{AB}, \overrightarrow{AC}\right)\right|, \text{ where :}$$

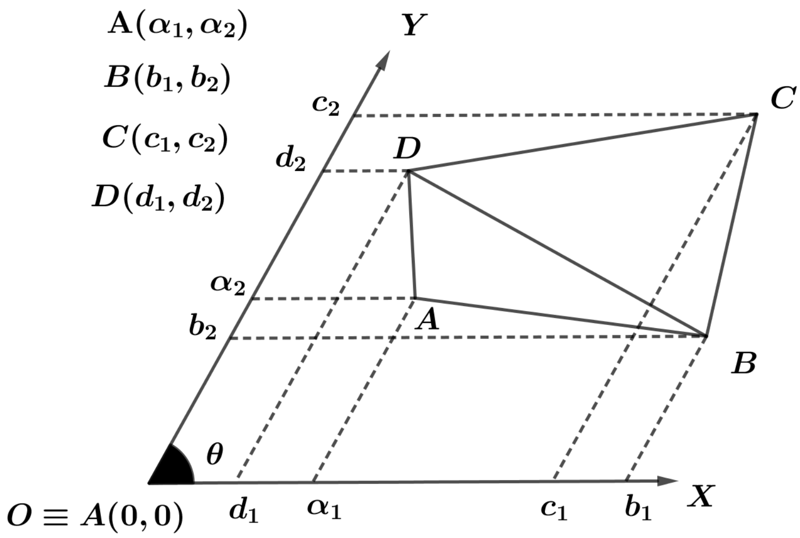
$A(\alpha_1, \alpha_2)$, $B(b_1, b_2)$, $C(c_1, c_2)$ and θ the angle of the system.

- Let $A(\alpha_1, \alpha_2)$, $B(b_1, b_2)$, $C(c_1, c_2)$, $D(d_1, d_2)$, then :

$$\textbf{(ABCD)}=\frac{\sin\theta}{2}\left|(\alpha_1 - c_1)(b_2 - d_2) - (b_1 - d_1)(\alpha_2 - c_2)\right|$$

or

$$\textbf{(ABCD)}=\frac{\sin\theta}{2}\left|\det\left(\overrightarrow{AC}, \overrightarrow{BD}\right)\right|$$



Proof

$$(\mathrm{ABD}) = \frac{\sin\theta}{2}\left|\begin{vmatrix} 1 & 1 & 1 \\ \alpha_1 & b_1 & d_1 \\ \alpha_2 & b_2 & d_2 \end{vmatrix}\right|$$
$$= \frac{\sin\theta}{2}|(b_1d_2 - d_1b_2) - (a_1d_2 - d_1a_2) + (a_1b_2 - b_1a_2)|$$

$$(\mathrm{BCD}) = \frac{\sin\theta}{2}\left|\begin{vmatrix} 1 & 1 & 1 \\ b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \end{vmatrix}\right|$$
$$= \frac{\sin\theta}{2}|(c_1d_2 - d_1c_2) - (b_1d_2 - d_1b_2) + (b_1c_2 - c_1b_2)|$$

Expanding the absolute values shall happen according to the repesentation of the area of the triangles that are constructed. When the points inside the determinant are written in counterclockwise direction, then the determinants are always positive.

Έτσι: $(ABCD) = (\mathrm{AB}D) + (\mathrm{B}CD) =$

$=\frac{\sin\theta}{2}(\alpha_1b_2 - b_1\alpha_2 - \alpha_1d_2 + d_1\alpha_2 + b_1c_2 - c_1b_2 - d_1c_2) =$

$=\frac{\sin\theta}{2}(\alpha_1(b_2 - d_2) - c_1(b_2 - d_2) + \alpha_2(d_1 - b_1) - c_2(d_1 - b_2) =$

$=\frac{\sin\theta}{2}((\alpha_1 - c_1)(b_2 - d_2) - (b_1 - d_1)(\alpha_2 - c_2))$

By choosing vertex A of the quadrilateral as an origin (we have the freedom of choosing any other vertex) and applying the formula with the absolute values, we get the formula :

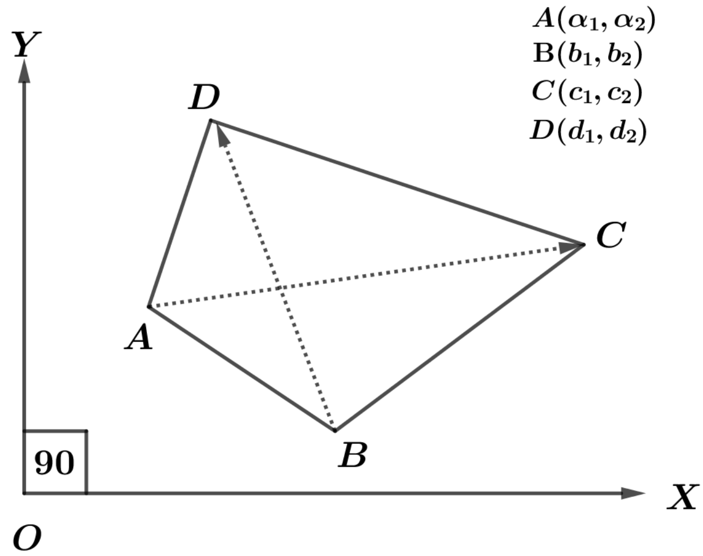
$(\mathrm{ABCD})=\frac{\sin\theta}{2}|(\alpha_1 - c_1)(b_2 - d_2) - (b_1 - d_1)(\alpha_2 - c_2)|$

or

$(\mathrm{ABCD})=\frac{\sin\theta}{2}\left|\det(\overrightarrow{\mathrm{AC}}, \overrightarrow{\mathrm{BD}})\right|$

Applications

I) Quadrilateral Area in orthogonal coordinate system



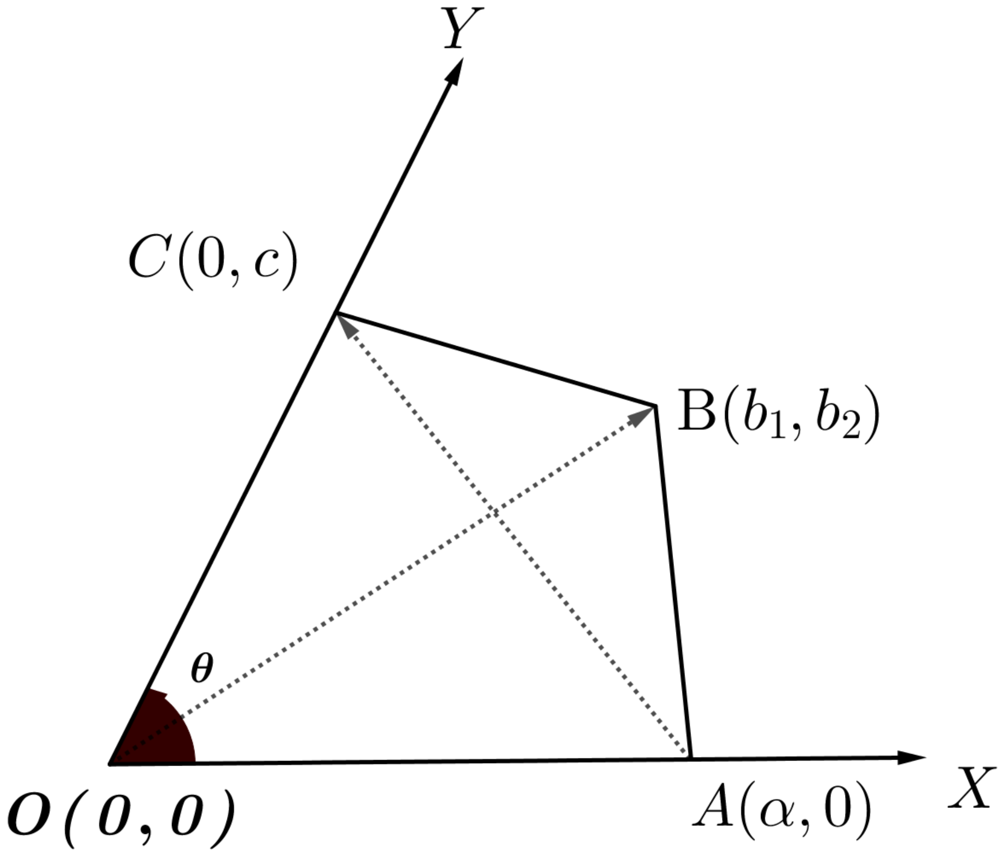
sinθ=sin90=1 , so

(ABCD)=$\frac{1}{2}\left|\mathbf{det}\,(\overrightarrow{\mathbf{AC}}, \overrightarrow{\mathbf{BD}})\right|$

or

(ABCD)=$\frac{1}{2}\left|(\boldsymbol{\alpha}_1 - \mathbf{c}_1)(\mathbf{b}_2 - \mathbf{d}_2) - (\mathbf{b}_1 - \mathbf{d}_1)(\boldsymbol{\alpha}_2 - \mathbf{c}_2)\right|$

II))Quadrilateral Area in plagiogonal system with axes two consecutive sides

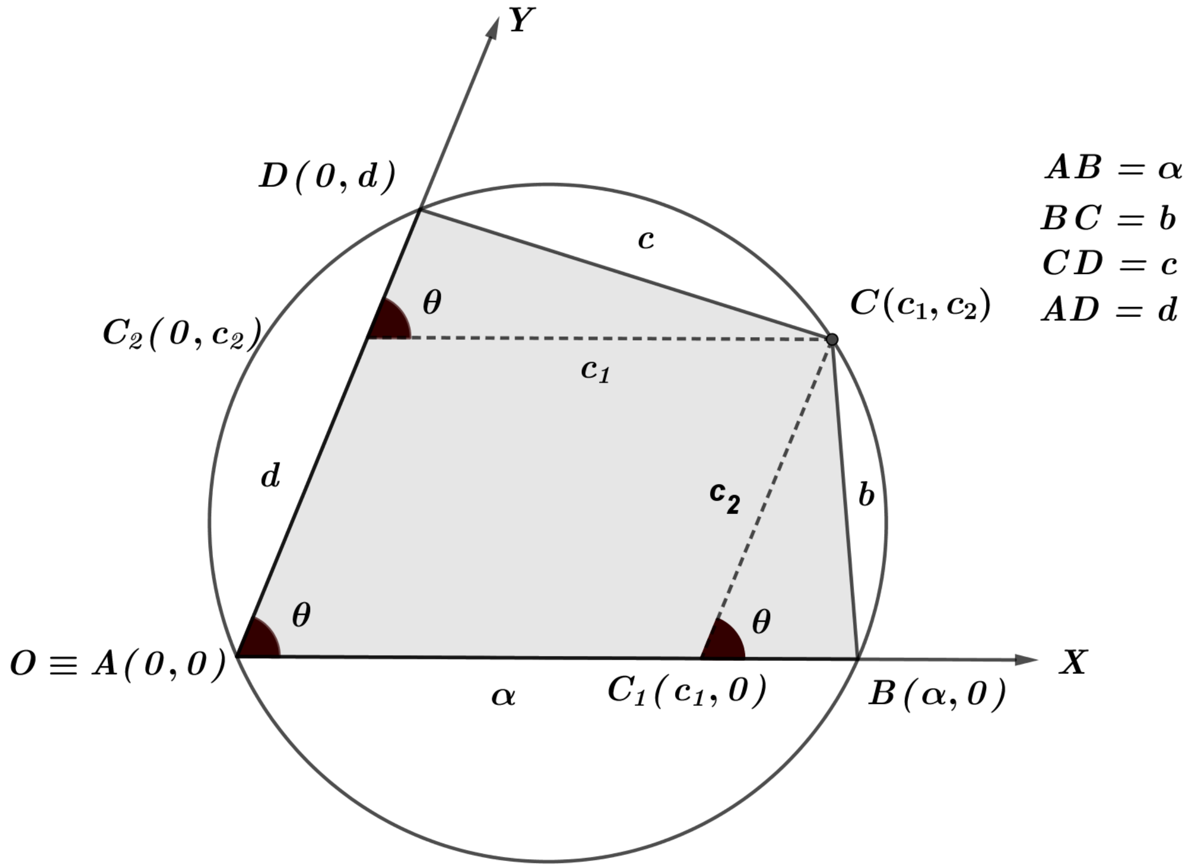


Let O(0,0) , A(α,0), B(b_1, b_2) , C(0,c) then:

$$(\mathrm{OABC}) = \frac{\sin\theta}{2}\left|\det\left(\overrightarrow{\mathrm{OB}}, \overrightarrow{\mathrm{AC}}\right)\right| = \frac{\sin\theta}{2}\left|\begin{vmatrix} b_1 & b_2 \\ -\alpha & c \end{vmatrix}\right| \Rightarrow$$

$$\boxed{(\mathbf{OABC}) = \frac{\mathbf{sin\theta}}{\mathbf{2}}|\boldsymbol{\alpha}\mathbf{b_2} + \boldsymbol{c}\boldsymbol{b_1}|}$$

III) Cyclic Quadrilateral Area



From a previous formula $(\mathrm{ABC}D) = \frac{\sin\theta}{2}|\alpha c_2 + dc_1|$ (1)

In triangle CC_1 , we have :$\frac{sinB}{CC_1} = \frac{sin\theta}{b} \Rightarrow \frac{sinB}{c_2} = \frac{sin\theta}{b} \Rightarrow c_2 = \frac{b\ sinB}{sin\theta}$ (2)

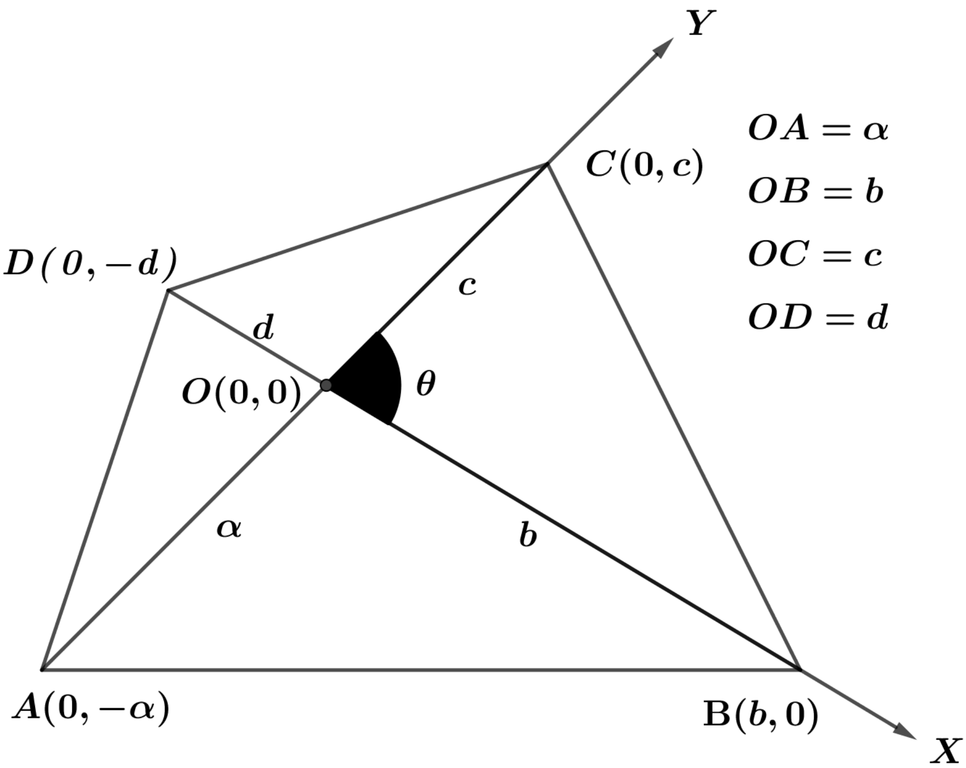
In triangle DCC_2 , we have:$\frac{sinD}{CC_2} = \frac{sin\theta}{c} \Rightarrow \frac{sinD}{c_1} = \frac{sin\theta}{c} \Rightarrow c_1 = \frac{c\ sinD}{sin\theta}$ (3)

Combining (1), (2), (3) :$(\mathrm{ABCD}) = \frac{\sin\theta}{2}(\alpha\frac{b\sin B}{\sin\theta} + d\frac{c\sin D}{\sin\theta}) = \frac{\sin B}{2}(\alpha b + cd)$

So:

$$\mathbf{(ABCD) = \frac{sinB}{2}(\alpha b + cd) = \frac{sinA}{2}(\alpha d + bc)}$$

IV) Quadrilateral Area in terms of the sidelengths of the diagonals and their angle



Let origin of the system be the intersection point O of the diagonals AC and BD, and angle θ of the system be the angle COB.
Then $O(0,0)$ and $A(-a,0), B(b,0), C(0,c), D(-d,0)$.

So:

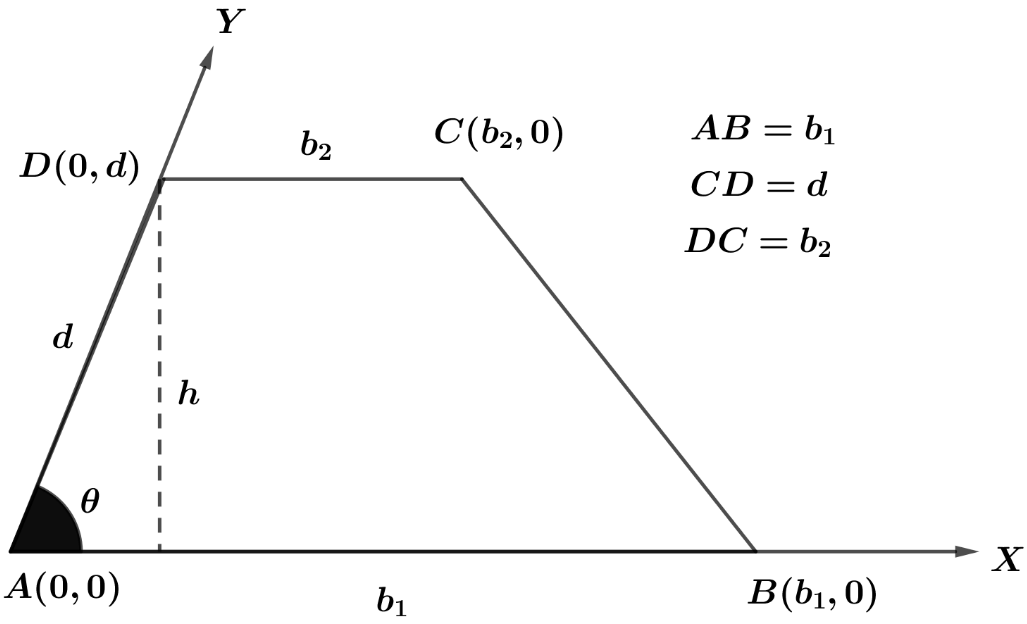
(ABCD)=$\frac{\sin\theta}{2}\left|\det\left(\overrightarrow{AC},\overrightarrow{BD}\right)\right| = \frac{\sin\theta}{2}\left|\begin{vmatrix} 0 & c+\alpha \\ -d-b & 0 \end{vmatrix}\right| = \frac{\sin\theta}{2}(\alpha+c)(b+d)$

⇒(ABCD)=$\frac{\sin\theta}{2}\mathbf{AC} * \mathbf{BD}$

V) Trapezoid Area

Let A be the angle of the plagiogonal system, $AB = b_1$ and $CD = b_2$ be the big and the small base of the trapezoid respectively.

Then, if $AD = d$, we have:



Let $A(0,0), B(b_1, 0), C(b_2, d), D(0, d)$, then:

(ABCD)=$\frac{\sin\theta}{2}\left|\det\left(\overrightarrow{AC}, \overrightarrow{BD}\right)\right| = \frac{\sin\theta}{2}\left|\begin{vmatrix} b_2 & d \\ -b_1 & d \end{vmatrix}\right| = \frac{\sin\theta}{2}(db_1 + db_2) \Rightarrow$

(ABCD)=$\frac{\sin\theta}{2}(db_1 + db_2) = \frac{\sin\theta}{2}d(b_1 + b_2)$

But $h = d\sin\theta$

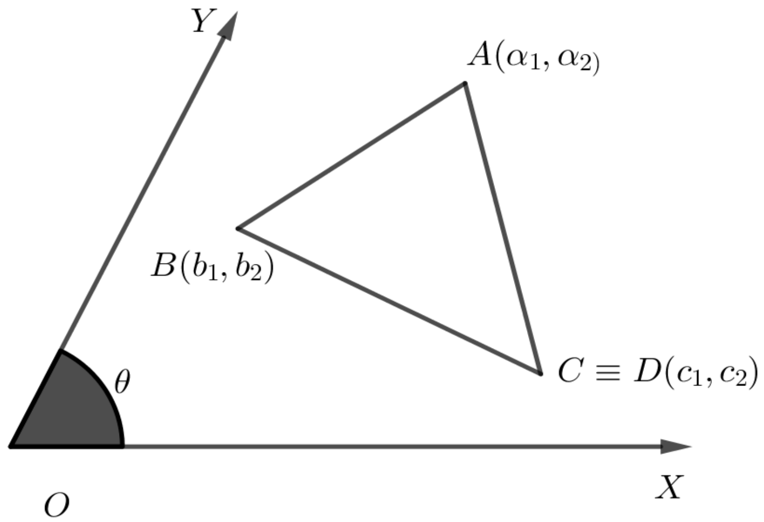
Finally:

(ABCD)=$\frac{1}{2}(\mathbf{b_1} + \mathbf{b_2})\mathbf{h}$

Remarks

- Using the formula of the quadrilateral area, we may find the formula of the triangle area, assuming that two vertices of the quadrilateral coincide.
  If the points C,D of the quadrilateral ABCD coincide, then:

$$(\mathrm{ABCD})=\frac{\sin\theta}{2}\left|\det(\overrightarrow{\mathrm{AC}},\overrightarrow{\mathrm{BD}})\right| = \frac{\sin\theta}{2}\left|\det(\overrightarrow{\mathrm{AC}},\overrightarrow{\mathrm{BC}})\right| = (\mathrm{ABC})$$



$$(\mathrm{ABC}) = \frac{\sin\theta}{2}\left|(\alpha_1-c_1)(b_2-c_2)-(b_1-c_1)(\alpha_2-c_2)\right| =$$

$$= \frac{\sin\theta}{2}\left|\alpha_1 b_2-\alpha_1 c_2-b_2c_1+c_1c_2-b_1\alpha_2+b_1c_2+\alpha_2c_1-c_1c_2\right| =$$

$$= \frac{\sin\theta}{2}\left\|\begin{matrix} 1 & 1 & 1 \\ \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \end{matrix}\right\|$$

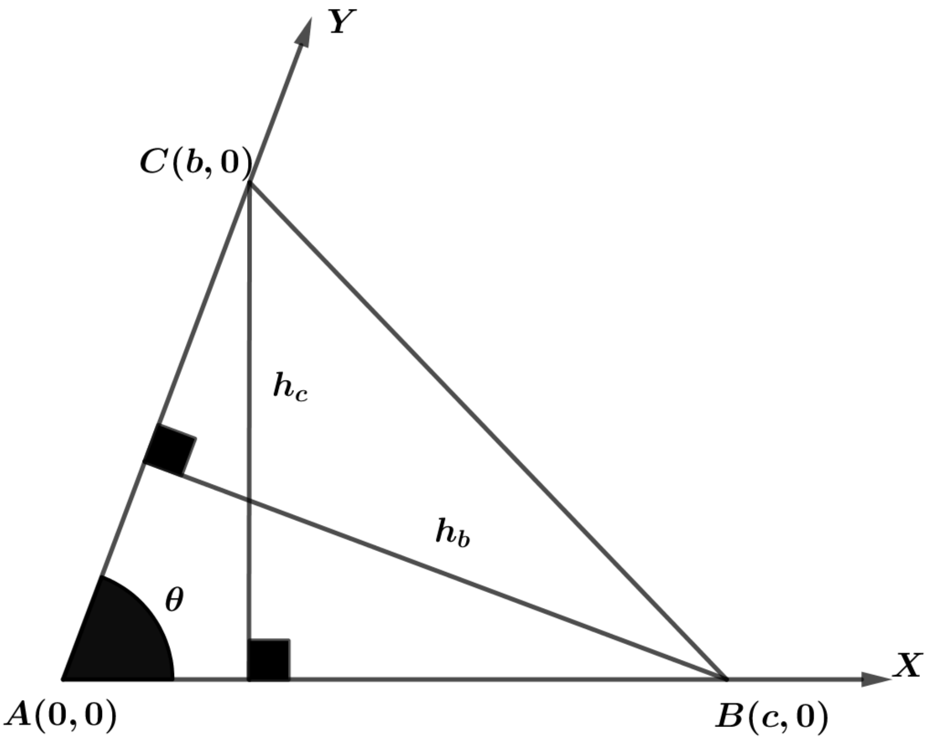
$$(\mathrm{ABC}) = \frac{\sin\theta}{2}\left|(\alpha_1-c_1)(b_2-c_2)-(b_1-c_1)(\alpha_2-c_2)\right| =$$

$$= \frac{\sin\theta}{2}\left\|\begin{matrix} \alpha_1-c_1 & \alpha_2-c_2 \\ b_1-c_1 & b_2-c_2 \end{matrix}\right\| = \frac{\sin\theta}{2}\left|\det(\overrightarrow{AC},\overrightarrow{BC})\right|$$

Finally:

$$\boxed{\mathbf{(ABC)} = \frac{\sin\theta}{2}\left|\det(\overrightarrow{AC},\overrightarrow{BC})\right| = \frac{\sin\theta}{2}\left\|\begin{matrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \alpha_1 & \mathbf{b_1} & \mathbf{c_1} \\ \alpha_2 & \mathbf{b_2} & \mathbf{c_2} \end{matrix}\right\|}$$

- If the two sides of the triangles are also axes of the plagiogonal system and vertex A is the origin of the system, then :
  A(0,0), B(c,0),C(0,b)

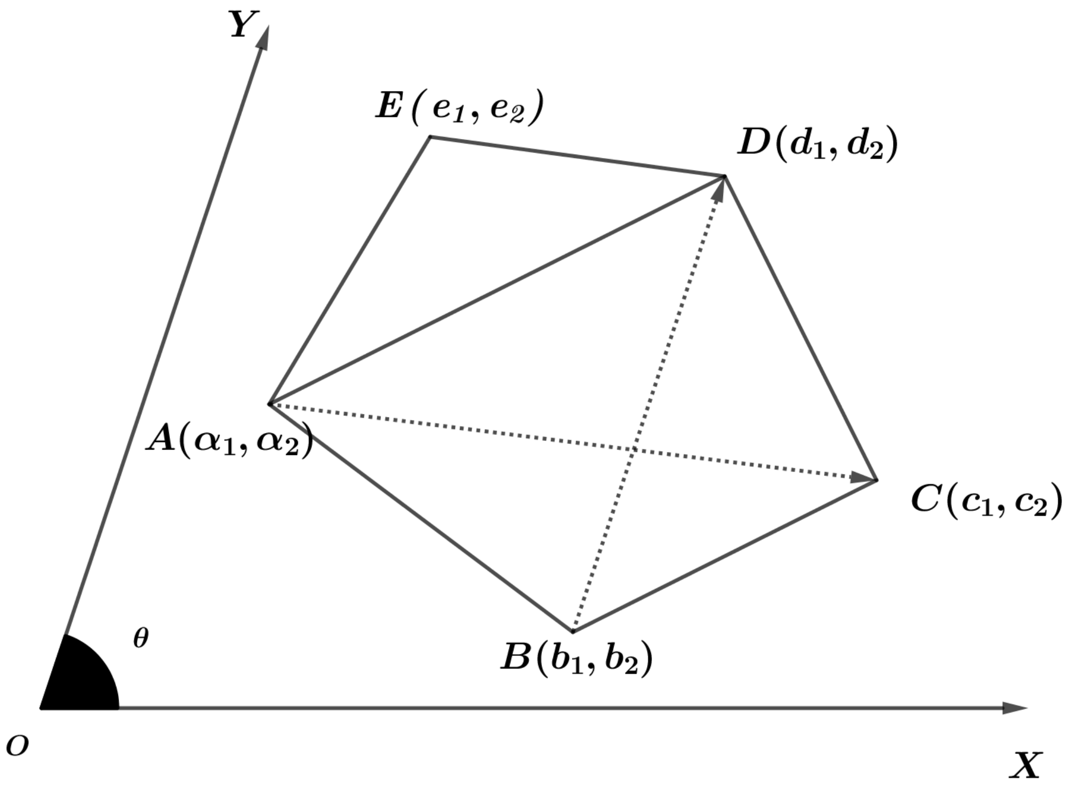


$$(ABC)=\frac{\sin A}{2}\left(\left|\det\left(\overrightarrow{AC},\overrightarrow{BD}\right)\right|\right)=\frac{\sin A}{2}\left|\begin{vmatrix}0 & b\\ -c & b\end{vmatrix}\right|=\frac{\sin A}{2}\left|\begin{vmatrix}0 & b\\ -c & b\end{vmatrix}\right|=\frac{\sin A}{2}bc$$

Finally

$$\mathbf{(ABC)}=\frac{\sin A}{2}bc=\frac{1}{2}bh_b=\frac{1}{2}ch_c$$

B) Pentagon Area



$$(ABCDE)=(ABCD)+(ADE)=\frac{\sin\theta}{2}\left(\left|\det\left(\overrightarrow{AC},\overrightarrow{BD}\right)\right|+\left|\det\left(\overrightarrow{AD},\overrightarrow{AE}\right)\right|\right)$$

$$(ABCDE)=\frac{\sin\theta}{2}\left\|\begin{vmatrix} c_1-\alpha_1 & c_2-\alpha_2 \\ d_1-b_1 & d_2-b_2 \end{vmatrix}\right\|+\frac{\sin\theta}{2}\left\|\begin{vmatrix} d_1-\alpha_1 & d_2-\alpha_2 \\ e_1-\alpha_1 & e_2-\alpha_2 \end{vmatrix}\right\|\Rightarrow$$

$$(ABCDE)=\frac{\sin\theta}{2}\left|(c_1-\alpha_1)(d_2-b_2)-(c_2-\alpha_2)(d_1-b_1)\right|$$
$$+\frac{\sin\theta}{2}\left|(d_1-\alpha_1)(e_2-\alpha_2)-(d_2-\alpha_2)(e_1-\alpha_1)\right|$$

With the proper orientation of the areas that represent these algebraic expressions, we expand the absolute values.

After calculations, we end up:

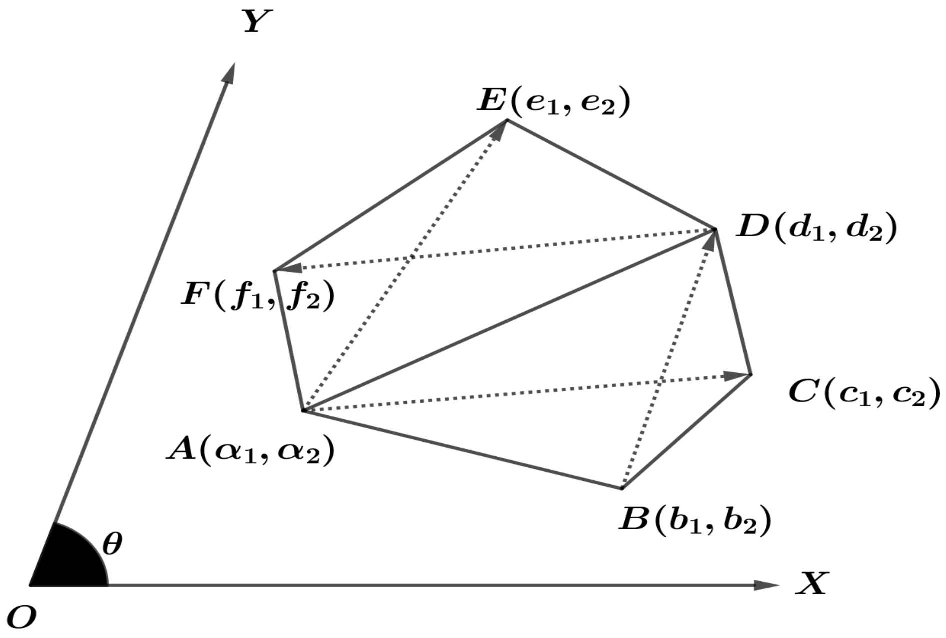
$$(\mathbf{ABCDE}) = \frac{\mathbf{sin\theta}}{\mathbf{2}}\left(\begin{vmatrix} \boldsymbol{\alpha_1} & \boldsymbol{\alpha_2} \\ \boldsymbol{b_1} & \mathbf{b_2} \end{vmatrix} + \begin{vmatrix} \mathbf{b_1} & \mathbf{b_2} \\ \mathbf{c_1} & \mathbf{c_2} \end{vmatrix} + \begin{vmatrix} \mathbf{c_1} & \mathbf{c_2} \\ \mathbf{d_1} & \mathbf{d_2} \end{vmatrix} + \begin{vmatrix} \mathbf{d_1} & \mathbf{d_2} \\ \mathbf{e_1} & \mathbf{e_2} \end{vmatrix} + \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} \\ \boldsymbol{\alpha_1} & \boldsymbol{\alpha_2} \end{vmatrix}\right)$$

Denote : $\begin{vmatrix} \mathrm{A} \\ \mathrm{B} \end{vmatrix} = \begin{vmatrix} \mathrm{a}_1 & \alpha_2 \\ \mathrm{b}_1 & \mathrm{b}_2 \end{vmatrix}$, $\begin{vmatrix} \mathrm{B} \\ \mathrm{C} \end{vmatrix} = \begin{vmatrix} \mathrm{b}_1 & \mathrm{b}_2 \\ \mathrm{c}_1 & \mathrm{c}_2 \end{vmatrix}$ $e.t.c$

Then the last formula can be easier recalled as:

$$(\mathbf{ABCDE}) = \frac{\mathbf{sin\theta}}{\mathbf{2}}\left(\begin{vmatrix} \mathbf{A} \\ \mathbf{B} \end{vmatrix} + \begin{vmatrix} \mathbf{B} \\ \mathbf{C} \end{vmatrix} + \begin{vmatrix} \mathbf{C} \\ \mathbf{D} \end{vmatrix} + \begin{vmatrix} \mathbf{D} \\ \mathbf{E} \end{vmatrix} + \begin{vmatrix} \mathbf{E} \\ \mathbf{A} \end{vmatrix}\right)$$

C) Hexagon Area



(ABCDEF)=$(\mathrm{ABCD}) + (\mathrm{ADEF}) = \frac{\sin\theta}{2}\left|\det(\overrightarrow{\mathrm{AC}}, \overrightarrow{\mathrm{BD}})\right| + \left|\det(\overrightarrow{\mathrm{AE}}, \overrightarrow{\mathrm{DF}})\right|$

With the proper orientation of the areas that represent these algebraic expressions, we expand the absolute values.

After calculations, we end up:

$$(\mathbf{ABCDEF}) =$$

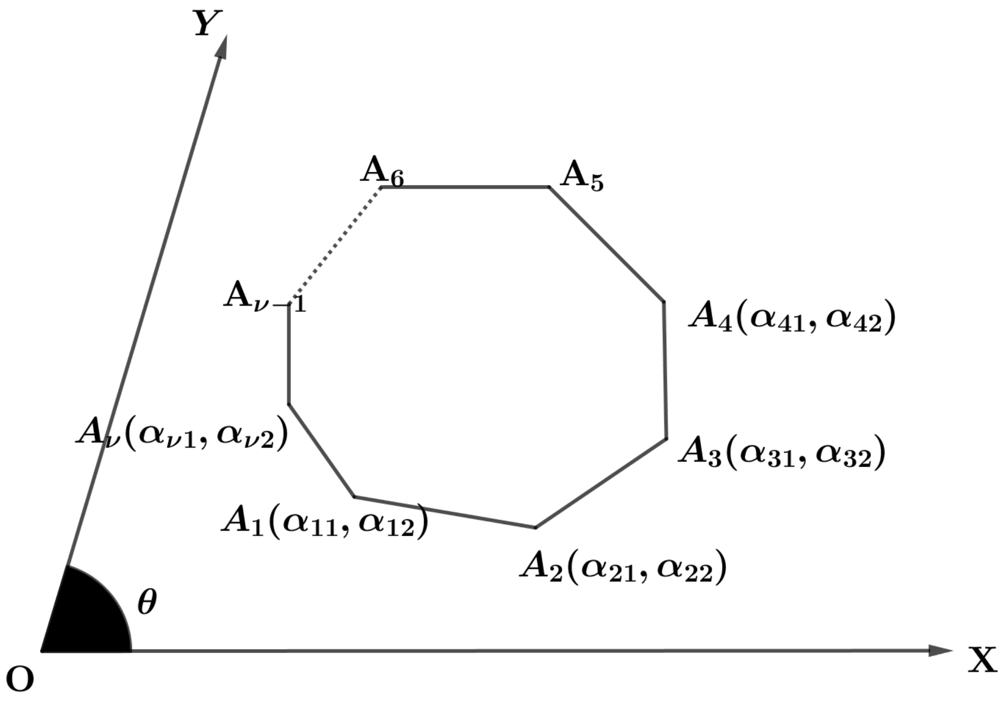
$$= \frac{\sin\theta}{2}\left(\begin{vmatrix} \alpha_1 & \alpha_2 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} + \begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix} + \begin{vmatrix} d_1 & d_2 \\ e_1 & e_2 \end{vmatrix} + \begin{vmatrix} e_1 & e_2 \\ f_1 & f_2 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 \\ \alpha_1 & \alpha_2 \end{vmatrix}\right)$$

Denote : $\begin{vmatrix} A \\ B \end{vmatrix} = \begin{vmatrix} a_1 & \alpha_2 \\ b_1 & b_2 \end{vmatrix}, \begin{vmatrix} B \\ C \end{vmatrix} = \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} e.t.c.$

Then the last formula can be easier recalled as:

$$(\mathbf{ABCDEF}) = \frac{\sin\theta}{2}\left(\begin{vmatrix} A \\ B \end{vmatrix} + \begin{vmatrix} B \\ C \end{vmatrix} + \begin{vmatrix} C \\ D \end{vmatrix} + \begin{vmatrix} D \\ E \end{vmatrix} + \begin{vmatrix} E \\ F \end{vmatrix} + \begin{vmatrix} F \\ A \end{vmatrix}\right)$$

D) Polygon area (n-gon)



Similarly we end up to the formula for the area of a n-gon in a **plagiogonal** coordinate system of same unit lengths.

$$(\mathbf{A_1A_2A_3 \ldots . A_{\nu-1}A_\nu}) =$$

$$= \frac{\sin\theta}{2}\left(\begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} + \begin{vmatrix} \alpha_{21} & \alpha_{22} \\ \alpha_{31} & \alpha_{32} \end{vmatrix} + \begin{vmatrix} \alpha_{31} & \alpha_{32} \\ \alpha_{41} & \alpha_{42} \end{vmatrix} + \cdots + \begin{vmatrix} \alpha_{\nu 1} & \alpha_{\nu 2} \\ \alpha_{11} & \alpha_{12} \end{vmatrix}\right)$$

Or by using the previously mentioned notation as

$$(\mathbf{A_1A_2A_3 \ldots . A_{\nu-1}A_\nu}) =$$

$$= \frac{\sin\theta}{2}\left(\begin{vmatrix} A_1 \\ A_2 \end{vmatrix} + \begin{vmatrix} A_2 \\ A_3 \end{vmatrix} + \begin{vmatrix} A_3 \\ A_4 \end{vmatrix} + \cdots + \begin{vmatrix} A_{\nu-1} \\ A_\nu \end{vmatrix} + \begin{vmatrix} A_\nu \\ A_1 \end{vmatrix}\right)$$

A few of the problems, in which the previous formulas where used are:

From the fb group : Romantics of Geometry (Ρομαντικοί της Γεωμετρίας)

- 2607 Thanos Kalogerakis
- 507 Stathis Koutras
- 2626 Thanasis Gakopoulos
- 2645 Thanasis Gakopoulos
- 2653 from another fb group
- 2659 Thanasis Gakopoulos
- 222 Vaggelis Stamatiadis
- 2671 Thanasis Gakopoulos
- 2678 Thanasis Gakopoulos (generalization)
- 2680 Brune Theorem
- 2702 Thanos Kalogerakis
- 2705 Thanasis Gakopoulos
- 2710 Arsalan Wares
- 2719 Arsalan Wares
- 2743 Thanasis Gakopoulos