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ABOUT PROBLEM JP.233

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1) Find the maximum and minimum possible value of:

$$\frac{1}{\sin^4 x + \cos^2 x} + \frac{1}{\cos^4 x + \sin^2 x}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution

We prove that $\sin^4 x + \cos^2 x = \cos^4 x + \sin^2 x \Leftrightarrow \sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x \Leftrightarrow$

$$\Leftrightarrow (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x) = \sin^2 x - \cos^2 x \Leftrightarrow \sin^2 x - \cos^2 x = \sin^2 x - \cos^2 x.$$

The problem returns to finding the maximum and the minimum of the expression

$$E(x) = \frac{2}{\sin^4 x + \cos^2 x}$$

We have $\frac{2}{\sin^4 x+\cos^2 x} = \frac{2}{\sin^4 x+1-\sin^2 x} = \frac{2}{t^2-t+1}$, *where* $t = \sin^2 x \in [0,1]$

We consider the function having the second degree $f:[0,1] \to \mathbb{R}, f(t) = t^2 - t + 1$

We have $f(0) = f(1) = 1 = \max$ *and* $f\left(\frac{1}{2}\right) = \frac{3}{4} = \min$

We obtain $\frac{3}{4} \le f(t) \le 1, \forall t \in [0,1]$, *wherefrom* $1 \le \frac{1}{f(t)} \le \frac{4}{3}$ *and so,* $2 \le \frac{2}{f(t)} \le \frac{8}{3}$

We obtain $2 \le E(x) \le \frac{8}{3}$. *It follows:*

$\max E(x) = \frac{8}{3}$ *and the maximum is attained for* $\sin^2 x = \frac{1}{2}$

$\min E(x) = 2$ *and the minimum is attained for* $\sin^2 x = 0$ *or* $\sin^2 x = 1$.

Remark: In the same way, we can propose:

1) Find all the values of the following expression:

$$\frac{\sin^2 x}{\cos^2 x + \tan^2 x} + \frac{\cos^2 x}{\sin^2 x + \cot^2 x}, \text{ where } x \in \left(0, \frac{\pi}{2}\right)$$

Proposed by Marin Chirciu – Romania

Solution: We prove that $\frac{\sin^2 x}{\cos^2 x + \tan^2 x} = \frac{\cos^2 x}{\sin^2 x + \cot^2 x} \Leftrightarrow \sin^4 x + \sin^2 x \cot^2 x =$

$$= \cos^4 x + \cos^2 x \tan^2 x \Leftrightarrow \sin^4 x + \cos^2 x = \cos^4 x + \sin^2 x \Leftrightarrow \sin^4 x - \cos^4 x =$$

$$= \sin^2 x - \cos^2 x \Leftrightarrow (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x) = \sin^2 x - \cos^2 x \Leftrightarrow$$

$$\Leftrightarrow \sin^2 x - \cos^2 x = \sin^2 x - \cos^2 x$$

The problem returns to finding all the values of the expression:

$$E(x) = \frac{2\sin^2 x}{\cos^2 x + \tan^2 x}, x \in \left(0, \frac{\pi}{2}\right)$$

We have $\frac{2\sin^2 x}{\cos^2 x + \tan^2 x} = \frac{2\left(1 - \frac{1}{1+\tan^2 x}\right)}{\frac{1}{1+\tan^2 x} + \tan^2 x} = \frac{2\tan^2 x}{\tan^4 x + \tan^2 x + 1} = \frac{2t}{t^2+t+1}$, *where* $t = \tan^2 x \in (0, \infty)$

We have $\frac{2t}{t^2+t+1} > 0, \forall t \in (0, \infty)$ *and* $\frac{2t}{t^2+t+1} \le \frac{2}{3}, \forall t \in (0, \infty) \Leftrightarrow (t-1)^2 \ge 0$, *obviously,*

with equality for $t = 1$.

We obtain $0 < \frac{2t}{t^2+t+1} \le \frac{2}{3}, \forall t \in (0, \infty)$, *wherefrom* $0 < E(x) < \frac{2}{3}$

It follows:

$\max E(x) = \frac{2}{3}$ *and the maximum is attained for* $\tan^2 x = 1 \Leftrightarrow x = \frac{\pi}{4}$

The set of the values of the expression from enunciation is $Im\, E = \left(0, \frac{2}{3}\right]$

Refferences:

Romanian Mathematical Magazine-Interactive Journal-**www.ssmrmh.ro**