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ABOUT PROBLEM 5399

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By Marin Chirciu – Romania

1) Let a, b, c be positive real numbers. Prove that:

$$\frac{2a+2b}{\sqrt{6a^2+4ab+6b^2}}+\frac{2b+2c}{\sqrt{6b^2+4bc+6c^2}}+\frac{2c+2a}{\sqrt{6c^2+4ca+6a^2}}\leq 3$$

Proposed by Angel Plaza – University of Las Palmas de Gran Canaria – Spain

Solution

We have $6a^2+4ab+6b^2\geq 4(a+b)^2\Leftrightarrow 2(a-b)^2\geq 0$***, obviously, with equality for*** $a=b$***. It follows*** $\sum\frac{2a+2b}{\sqrt{6a^2+4ab+6b^2}}\leq\sum\frac{2a+2b}{\sqrt{4(a+b)^2}}=\sum\frac{2a+2b}{2(a+b)}=3$

We deduce that the inequality from enunciation holds, with equality if and only if $a=b=c$.

Remark. The problem can be developed.

2) If $a, b, c>0$ and $n\geq 1$, prove that:

$$\frac{a+b}{\sqrt{na^2+(16-2n)ab+nb^2}}+\frac{b+c}{\sqrt{nb^2+(16-2n)bc+nc^2}}+\frac{c+a}{\sqrt{nc^2+(16-2n)ca+na^2}}\leq\frac{3}{2}$$

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Solution

We have $na^2 + (4-2n)ab + nb^2 \geq (a+b)^2 \Leftrightarrow (n-1)(a-b)^2 \geq 0$*, obviously, with equality for* $n = 1$ *or* $a = b$.

It follows $\sum \frac{a+b}{\sqrt{na^2+(4-2n)ab+nb^2}} \leq \sum \frac{a+b}{\sqrt{(a+b)^2}} = \sum \frac{a+b}{a+b} = 3$

We deduce that the inequality from enunciation holds, with equality if and only if $n = 1$ *or* $a = b = c$.

3) If $a, b, c > 0$ and $n \geq 4$, prove that:

$$\frac{a+b}{\sqrt{na^2+(16-2n)ab+nb^2}} + \frac{b+c}{\sqrt{nb^2+(16-2n)bc+nc^2}} + \frac{c+a}{\sqrt{nc^2+(16-2n)ca+na^2}} \leq \frac{3}{2}$$

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Solution

We have $na^2 + (16-2n)ab + nb^2 \geq 4(a+b)^2 \Leftrightarrow (n-4)(a-b)^2 \geq 0$*, obviously, with equality for* $n = 4$ *or* $a = b$.

It follows $\sum \frac{a+b}{\sqrt{na^2+(16-2n)ab+nb^2}} \leq \sum \frac{a+b}{\sqrt{4(a+b)^2}} = \sum \frac{a+b}{2(a+b)} = \frac{3}{2}$

We deduce that the inequality from enunciation holds, with equality if and only if $n = 4$ *or* $a = b = c$.

4) If $a, b, c > 0$ and $n \geq 9$, prove that:

$$\frac{a+b}{\sqrt{na^2+(36-2n)ab+nb^2}} + \frac{b+c}{\sqrt{nb^2+(36-2n)bc+nc^2}} + \frac{c+a}{\sqrt{nc^2+(36-2n)ca+na^2}} \leq 1$$

Proposed by Marin Chirciu – Romania

Solution

We have $na^2 + (36-2n)ab + nb^2 \geq 9(a+b)^2 \Leftrightarrow (n-9)(a-b)^2 \geq 0$*, obviously, with equality for* $n = 4$ *or* $a = b$*. It follows*

$$\sum \frac{a+b}{\sqrt{na^2+(36-2n)ab+nb^2}} \le \sum \frac{a+b}{\sqrt{9(a+b)^2}} = \sum \frac{a+b}{3(a+b)} = 1$$

We deduce that the inequality from enunciation holds, with equality if and only if $n = 9$ or

$$a = b = c.$$

Remark.The inequality can be generalized:

5) If $a, b, c > 0$ and $n \ge k^2 > 0$, prove that:

$$\frac{a+b}{\sqrt{na^2+(4k^2-2n)ab+nb^2}} + \frac{b+c}{\sqrt{nb^2+(4k^2-2n)bc+nc^2}} + \frac{c+a}{\sqrt{nc^2+(4k^2-2n)ca+na^2}} \le \frac{3}{k}$$

Proposed by Marin Chirciu – Romania

Solution

We have $na^2 + (4k^2 - 2n)ab + nb^2 \ge k^2(a+b)^2 \Leftrightarrow (n-k^2)(a-b)^2 \ge 0$, obviously,

with equality for $n = 4$ or $a = b$.

It follows $\sum \frac{a+b}{\sqrt{na^2+(4k^2-2n)ab+nb^2}} \le \sum \frac{a+b}{\sqrt{k^2(a+b)^2}} = \sum \frac{a+b}{k(a+b)} = \frac{3}{k}$

We deduce that the inequality from enunciation holds, with equality if and only if $n = k^2$

or $a = b = c$.