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ROMANIAN MATHEMATICAL MAGAZINE
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ABOUT CALCULUS OF SOME AMAZING LIMITS

By Florentin Vișescu-Romania

Lemma

Let be $q \in \mathbb{N}^*$ and $f: [0: q] \to [0, \infty)$ a continuous function, then:

$$\lim_{n\to\infty} \prod_{i=1}^{2n} \left(1 + \frac{1}{n} f\left(\frac{i}{n}\right)\right)$$

Proof.

$$\prod_{i=1}^{qn} \left(1 + \frac{1}{n} f\left(\frac{i}{n}\right)\right) = \prod_{i=1}^{qn} e^{\ln\left(1+\frac{1}{n}f\left(\frac{i}{n}\right)\right)} = e^{\sum_{i=1}^{qn} \ln\left(1+\frac{1}{n}f\left(\frac{i}{n}\right)\right)}$$

We prove that $x - \frac{x^2}{2} \le \ln(1 + x) \le x, \forall x \in [0, \infty)$

Consider $q: [0, \infty) \to \mathbb{R}, g(x) = \ln(1 + x) - x$

$$g'(x) = \frac{1}{1 + x} - 1 = -\frac{x}{1 + x} \le 0, \forall x \in [0, \infty) \Rightarrow$$

$q \searrow$ **on** $[0, \infty) \Rightarrow g(x) \le g(0), \forall x \in [0, \infty)$

or $\ln(1 + x) - x \le 0, \forall x \in [0, \infty)$

consider $h: [0, \infty) \to \mathbb{R}, h(x) = \ln(1 + x) - x + \frac{x^2}{2}$

$$h'(x)=\frac{1}{1+x}-1+x=\frac{1-1-x+x+x^2}{1+x}=\frac{x^2}{1+x}\geq 0$$

$\forall x\in(0,\infty)\Rightarrow h\nearrow$ on $[0,\infty)\Rightarrow h(x)\geq h(0),\forall x\in[0,\infty)$ or $\ln(1+x)\geq x-\frac{x^2}{2}$,

$$\forall x\in[0,\infty)$$

So, $x-\frac{x^2}{2}\leq \ln(1+x)\leq x,\forall x\in[0,\infty)$

We choose $x=\frac{1}{n}f\left(\frac{i}{n}\right);i,n\in\mathbb{N}^*\Rightarrow\frac{1}{n}f\left(\frac{i}{n}\right)-\frac{1}{2}\left(\frac{1}{n}f\left(\frac{i}{n}\right)\right)^2\leq\ln\left(1+\frac{1}{n}f\left(\frac{i}{n}\right)\right)\leq\frac{1}{n}f\left(\frac{i}{n}\right)$

$$\forall i,n\in\mathbb{N}^*$$

$$\sum_{i=1}^{qn}\frac{1}{n}f\left(\frac{i}{n}\right)-\frac{1}{n}\sum_{i=1}^{qn}\left(\frac{1}{n}f\left(\frac{i}{n}\right)\right)^2\leq\sum_{i=1}^{qn}\ln\left(1+\frac{1}{n}f\left(\frac{i}{n}\right)\right)\leq\sum_{i=1}^{qn}\frac{1}{n}f\left(\frac{i}{n}\right)$$

$$\lim_{n\to\infty}\sum_{i=1}^{qn}\frac{1}{n}f\left(\frac{i}{n}\right)=\int_0^q f(x)\,dx$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{qn}\frac{1}{n}f^2\left(\frac{i}{n}\right)=\lim_{n\to\infty}\frac{1}{n}\cdot\lim_{n\to\infty}\sum_{i=1}^{qn}\frac{1}{n}f^2\left(\frac{i}{n}\right)=0\cdot\int_0^q f^2(x)\,dx=0$$

Then $\lim_{n\to\infty}\sum_{i=1}^{qn}\ln\left(1+\frac{1}{n}f\left(\frac{i}{n}\right)\right)=\int_0^q f(x)\,dx$

and so, $\lim_{n\to\infty}\prod_{i=1}^{qn}\left(1+\frac{1}{n}f\left(\frac{i}{n}\right)\right)=e^{\int_0^q f(x)dx}$

Then:

1)

$$\lim_{n\to\infty}\prod_{i=1}^{qn}\left(1+\frac{1}{n}\left(\frac{i}{n}\right)^p\right)=e^{\frac{q^{p+1}}{1+p}};q,p\in\mathbb{N}^*$$

2)

$$\lim_{n\to\infty}\prod_{i=1}^{qn}\left(1+\sqrt[p]{\frac{n+i}{n^{p+1}}}\right)=e^{\frac{p}{p+1}\left(\sqrt[p]{(q+1)^{p+1}}-1\right)};q,p\in\mathbb{N}^*;p\geq 2$$

3)

$$\lim_{n\to\infty}\prod_{i=1}^{qn}\left(1+\frac{\sqrt[n]{q^i}}{n}\right)=e^{\frac{q^q-1}{\ln 2}};q\in\mathbb{N}^*$$

4)

$$\lim_{n\to\infty}\sum_{i=1}^{qn}\left(1+\frac{1}{n}\cos^2\frac{i}{2n}\right)=\left(\sqrt{e}\right)^{q+\sin q};q\in\mathbb{N}^*$$

5)

$$\lim_{n\to\infty}\prod_{i=1}^{n^2}\left(1+\frac{i^p}{n^{2p+2}}\right)=e^{\frac{1}{p+1}};p,q\in\mathbb{N}^*$$