R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

UP.252. If $m, n, t > 0$ then in ΔABC the following relationship holds:

$$m^3\left(\tan\frac{A}{2}\cdot\tan\frac{B}{2}\right)^4+n^3\left(\tan\frac{B}{2}\tan\frac{C}{2}\right)^4+t^3\left(\tan\frac{C}{2}\tan\frac{A}{2}\right)^4\ge\frac{mntr^2}{s^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

Solution by proposers

According to AM-GM inequality:

$$V=m^3\left(\tan\frac{A}{2}\tan\frac{B}{2}\right)^4+n^3\left(\tan\frac{B}{2}\tan\frac{C}{2}\right)^4+t^3\left(\tan\frac{C}{2}\tan\frac{A}{2}\right)^4\ge$$

$$\ge 3\cdot\sqrt[3]{m^3n^3t^3\left(\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\right)^8}=3mn\left(\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\right)^2\cdot\sqrt[3]{\left(\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\right)^2}\quad (1)$$

But $\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}=\frac{r}{s}$ ***(2)***

$$\text{So, } V\ge 3mnt\cdot\frac{r^2}{s^2}\sqrt[3]{\frac{r^2}{s^2}}\overset{Mitrinovic}{\ge}3mnt\cdot\frac{r^2}{s^2}\sqrt[3]{\frac{r^2}{\left(3\sqrt{3}r\right)^2}}=9mnt-\frac{r^2}{s^2}\cdot\frac{1}{3}=$$

$=mnt\cdot\frac{r^2}{s^2}$ ***q.e.d with equality ⇔ the triangle is equilateral,*** $m=n=t$.