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UP.241. Let be $n \in \mathbb{N}; n \geq 3; m \geq 0; A_1A_2 \dots A_n$; a convexe polygon inscribed in a circle with radius R. If $a_k = A_kA_{k+1}; k \in \overline{1,n}; A_{n+1} = A_1$; s – semiperimeter then:

$$\sum_{k=1}^{n} \frac{1}{a_k^m} \geq \frac{n}{2^m R^m \sin^m \frac{\pi}{n}}$$

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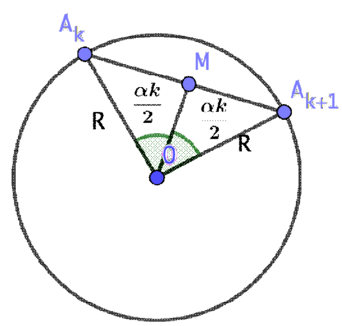

$$\Delta OMA_{k+1}: MA_{k+1} = \frac{a_k}{2}$$

$$\sin \frac{\alpha_k}{2} = \frac{MA_{k+1}}{OA_{k+1}} = \frac{\frac{a_k}{2}}{R} = \frac{a_k}{2R}$$

$$a_k = 2R \sin \frac{\alpha k}{2}$$

$$\sum_{k=1}^{n}\frac{1}{a_k^m}\overset{J.Radon}{>}\frac{(1+1+\cdots+1)^{m+1}}{(a_1+a_2+\cdots+a_n)^m}=\frac{n^{m+1}}{(2s)^m}=\frac{n^{m+1}}{2^m\cdot s^m}$$

$$s=\frac{a_1+a_2+\cdots+a_n}{2}=\frac{2R\left(\sin\frac{\alpha_1}{2}+\sin\frac{\alpha_2}{2}+\cdots+\sin\frac{\alpha_n}{2}\right)}{2}=$$

$$=R\left(\sin\frac{\alpha_1}{2}+\sin\frac{\alpha_2}{2}+\cdots+\sin\frac{\alpha_n}{2}\right)$$

But $\alpha_1+\alpha_2+\cdots+\alpha_n=2\pi\Rightarrow\frac{\alpha_1+\alpha_2+\cdots+\alpha_n}{2}=\frac{2\pi}{2}=\pi$

Let be $f(x)=\sin x\Rightarrow f''(x)=\cos x\Rightarrow f''(x)=-\sin x<0\Rightarrow f\to$ ***concave***

We apply Jensen's inequality to function f***:***

$$\frac{f\left(\frac{\alpha_1}{2}\right)+f\left(\frac{\alpha_2}{2}\right)+\cdots+f\left(\frac{\alpha_n}{2}\right)}{n}\le f\left(\frac{\frac{\alpha_1}{2}+\frac{\alpha_2}{2}+\cdots+\frac{\alpha_n}{2}}{n}\right)\Rightarrow$$

$$\Rightarrow\sin\frac{\alpha_1}{2}+\sin\frac{\alpha_2}{2}+\cdots+\sin\frac{\alpha_n}{2}\le n\sin\frac{\pi}{n}$$

So, $\sum_{k=1}^{n}\frac{1}{a_k^m}\ge\frac{n^{m+1}}{2^m\cdot R^m\cdot\sin^m\frac{\pi}{m}\cdot n^m}=\frac{n}{2^m\cdot R^m\cdot\sin^m\frac{\pi}{n}}$