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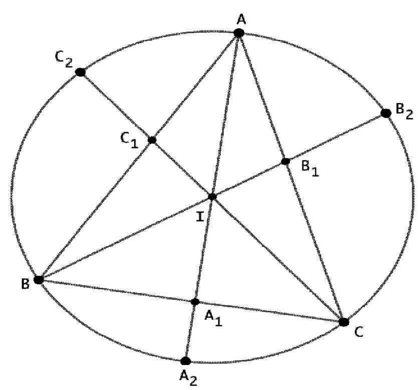
SP. 242 Let w'_a, w'_b, w'_c be the circumpedal extensions of cevians of incentre in ΔABC. Prove that:

$$w'_a + w'_b + w'_c \geq 6\sqrt[3]{\frac{a^2b^2c^2}{(a+b)(b+c)(c+a)}}$$

Proposed by Daniel Sitaru-Romania

Solution by proposer



$$AA_1 = w_a, BB_1 = w_b, CC_1 = w_c, AA_1 \cap BB_1 \cap CC_1 = \{I\}$$

$$\frac{A_1B}{A_1C} = \frac{c}{b} \Rightarrow \frac{A_1B}{A_1B + A_1C} = \frac{c}{b+c}$$

$$\frac{A_1B}{a} = \frac{c}{b+c} \Rightarrow A_1B = \frac{ac}{b+c}; A_1C = \frac{ab}{b+c}$$

$$\rho(A_1) = BA_1 \cdot A_1C = AA_1 \cdot A_1A_2$$

$$\frac{ac}{b+c} \cdot \frac{ab}{b+c} = w_a \cdot A_1A_2$$

$$A_1A_2 = \frac{a^2bc}{w_a(b+c)^2}$$

$$w'_a = AA_2 = AA_1 + A_1A_2 = w_a + \frac{a^2bc}{w_a(b+c)^2}$$

Analogous:

$$w'_b = w_b + \frac{ab^2c}{w_b(a+c)^2}; w'_c = w_c + \frac{abc^2}{w_c(b+a)^2}$$

By summing:

$$w'_a + w'_b + w'_c = w_a + \frac{a^2bc}{w_a(b+c)^2} + w_b + \frac{ab^2c}{w_b(a+c)^2} + w_c + \frac{abc^2}{w_c(b+a)^2} \geq$$

$$\overset{AM-GM}{\geq} 6\sqrt[6]{w_aw_bw_c \cdot \frac{a^2bc \cdot ab^2c \cdot abc^2}{w_aw_bw_c(b+c)^2(a+c)^2(b+a)^2}} =$$

$$= 6\sqrt[6]{\frac{a^4b^4c^4}{(a+b)^2(b+c)^2(c+a)^2}} = 6\sqrt[3]{\frac{a^2b^2c^2}{(a+b)(b+c)(c+a)}}$$