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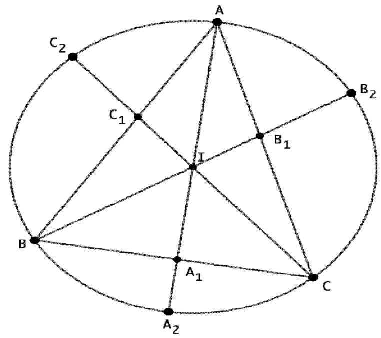
SP.241 Let w'_a, w'_b, w'_c be the circumpedal extensions of cevian of incentre in ΔABC. Prove that:

$$w'_a w'_b w'_c \geq \frac{8a^2b^2c^2}{(a+b)(b+c)(c+a)}$$

Proposed by Daniel Sitaru-Romania

Solution by proposer



$$AA_1 = w_a; BB_1 = w_b; CC_1 = w_c; AA_1 \cap BB_1 \cap CC_1 = \{I\}$$

$$\frac{A_1B}{A_1C} = \frac{c}{b} \Rightarrow \frac{A_1B}{A_1B + A_1C} = \frac{c}{b+c}$$

$$\frac{A_1B}{a} = \frac{c}{b+c} \Rightarrow A_1B = \frac{ac}{b+c}; A_1C = \frac{ab}{b+c}$$

$$\rho(A_1) = BA_1 \cdot A_1C = AA_1 \cdot A_1A_2$$

$$\frac{ac}{b+c}\cdot\frac{ab}{b+c}=w_a\cdot A_1A_2$$

$$A_1A_2=\frac{a^2bc}{w_a(b+c)^2}$$

$$w'_a=AA_2=AA_1+A_1A_2=w_a+\frac{a^2bc}{w_a(b+c)^2}\overset{AM-GM}{\geq}$$

$$\geq 2\sqrt{w_a\cdot\frac{a^2bc}{w_a(b+c)^2}}=\frac{2a}{b+c}\sqrt{bc}$$

Analogous:

$$w'_b=\frac{2b}{c+a}\sqrt{ca};w'_c=\frac{2c}{a+b}\sqrt{ab}$$

By multiplying:

$$w'_aw'_bw'_c\geq\frac{2a}{b+c}\sqrt{bc}\cdot\frac{2b}{c+a}\sqrt{ca}\cdot\frac{2c}{a+b}\sqrt{ab}=$$

$$=\frac{8abc\cdot abc}{(a+b)(b+c)(c+a)}\geq\frac{8a^2b^2c^2}{(a+b)(b+c)(c+a)}$$