PROPOSED PROBLEM

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Let $f_n = \left(1+\frac{1}{n}\right)^n ((2n-1)!!F_n)^{\frac{1}{n}}$. Find $\lim_{n\to\infty}(f_{n+1}-f_n)$ where F_n denotes the n^{th} Fibonacci number (given by $F_0 = 0, F_1 = 1$, and by $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 1$.

Solution by Soumitra Mandal - Chandar Nagore - India.

$$\lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!F_n}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{(2n-1)!!F_n}{n^n}}$$

$$\overset{\text{Cauchy D'Alembert}}{=} \lim_{n\to\infty} \left\{ \frac{F_{n+1}}{F_n} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{\frac{(2n+2)!}{2^{n+1}(n+1)!}}{\frac{(2n)!}{2^n n!}} \cdot \frac{1}{n+1} \right\} =$$

$$= \frac{\varphi}{e} \lim_{n\to\infty} \frac{2n+1}{n+1} = \frac{2\varphi}{e}$$

Let $x_n = \frac{\sqrt[n+1]{(2n+1)!!F_{n+1}}}{\sqrt[n]{(2n-1)!!F_n}}$ for all $n \in \mathbb{N}$. So,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \left(\frac{\sqrt[n+1]{(2n+1)!!F_{n+1}}}{n+1} \cdot \frac{n}{\sqrt[n]{(2n-1)!!F_n}} \cdot \left(1+\frac{1}{n}\right) \right) = 1$$

$$\therefore \frac{\frac{(1+\frac{1}{n+1})^{n+1}}{(1+\frac{1}{n})^n} x_n - 1}{\ln x_n} = \frac{\frac{(1+\frac{1}{n+1})^{n+1}}{(1+\frac{1}{n})^n} x_n - 1}{\ln\left\{ \frac{(1+\frac{1}{n+1})^{n+1}}{(1+\frac{1}{n})^n} x_n \right\}} \text{ tends to } 1 \, \frac{(1+\frac{1}{n+1})^{n+1}}{(1+\frac{1}{n})^n} x_n \to 1$$

$$\therefore \lim_{n\to\infty} x_n^n = \lim_{n\to\infty} \left(\frac{(2n+1)!!F_{n+1}}{(2n-1)!!F_n} \cdot \frac{1}{\sqrt[n+1]{(2n-1)!!F_{n+1}}} \right)$$

$$= \lim_{n\to\infty} \left(\frac{F_{n+1}}{F_n} \cdot \frac{2n+1}{n+1} \cdot \frac{n+1}{\sqrt[n+1]{(2n+1)!!F_{n+1}}} \right) = \left(\varphi \cdot 2 \cdot \frac{e}{2\varphi} \right) = e$$

$$\therefore \lim_{n\to\infty} (f_{n+1} - f_n) = \lim_{n\to\infty} \left(\left(1+\frac{1}{n+1}\right)^{n+1} \sqrt[n+1]{(2n+1)!!F_{n+1}} - \left(1+\frac{1}{n}\right)^n \sqrt[n]{(2n-1)!!F_n} \right)$$

$$= \lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)^n \cdot \frac{\sqrt[n]{(2n-1)!!F_n}}{n} \cdot \frac{\frac{(1+\frac{1}{n+1})^{n+1}}{(1+\frac{1}{n})^n} x_n - 1}{\ln x_n} \cdot \ln x_n^n \right) = e \cdot \frac{2\varphi}{e} \cdot \ln e = 2\varphi$$

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