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Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{n^3}\sum_{1\le i<j\le n}\left((-1)^{i+j}\cdot i\cdot j\right)\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Adrian Popa-Romania, Solution 2 by Naren Bhandari-Bajura-Nepal, Solution 3 by Remus Florin Stanca-Romania, Solution 4 by Florentin Vișescu-Romania

Solution 1 by Adrian Popa-Romania

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{n^3}\sum_{1\le i<j\le n}\left((-1)^{i+j}\cdot i\cdot j\right)\right)$$

$$= \frac{1}{2}\lim_{n\to\infty}\left[\frac{1}{n^3}(-1+2-3+4-5+6-\dots+(-1)^n n)^2-\sum_{k=1}^{n}k^2\right]$$

$\therefore$***We've used the following relationship:*** $(a+b)^2 = a^2+2ab+b^2 \Rightarrow$

$$\Rightarrow 2ab = (a+b)^2-a^2-b^2 \Rightarrow ab = \frac{(a+b)^2-a^2-b^2}{2}\ \therefore$$

$$2\Omega = \underbrace{\lim_{n\to\infty}\frac{\left(\overbrace{-1+2}^{1}-\overbrace{3+4}^{1}-\dots+(-1)^n n\right)^2}{n^3}}_{L_1}-\underbrace{\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k^2}{n^2}}_{L_2}$$

We evaluate L_1 using sandwich theorem:

$$\frac{\overbrace{1+1+\cdots+1}^{\left[\frac{n}{2}\right]\ times}-n}{\underset{\underset{0}{\downarrow}}{n^3}}<L_1<\frac{\frac{n}{2}\ times}{\underset{\underset{0}{\downarrow}}{n^3}}\Rightarrow L_1\to 0$$

We calculate L_2:

$$L_2=\int_0^1 x^2\,dx=\left.\frac{x^3}{3}\right|_0^1=\frac{1}{3}$$

$$2\Omega=0-\frac{1}{3}=-\frac{1}{3}\Rightarrow\Omega=-\frac{1}{6}$$

Solution 2 by Naren Bhandari-Bajura-Nepal

Denote:

$$L=\lim_{n\to\infty}\frac{1}{n^3}\sum_{1\le i<j\le n}(-1)^{i+j}\,ij=\lim_{n\to\infty}\sum_{j=1}^{n}\sum_{i\ne 1}^{n}(-1)^{i+j}ij$$

$$=\lim_{n\to\infty}\frac{1}{n^3}\sum_{j=1}^{n}\sum_{i=j+1}^{n}(-1)^{i+j}=\lim_{n\to\infty}\frac{1}{n^3}\left(\sum_{j=1}^{n}(-1)^j j\right)\left(\sum_{i=1}^{n}(-1)^i j\right)$$

$$-\lim_{n\to\infty}\frac{1}{n^3}\left(\sum_{j=1}^{n}\sum_{i=1}^{n}(-1)^j j((-1)^i i)\right)$$

$$=\lim_{n\to\infty}\frac{1}{n^3}\left(\sum_{k=1}^{n}(2k+1)-(2k)\right)^2-\lim_{n\to\infty}\frac{1}{n^3}\sum_{j=1}^{n}\sum_{i=j+1}^{n}(-1)^{j+1}ij$$

$$-\lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}k^2\ (i=j=k)$$

$$=\lim_{n\to\infty}\frac{n^2}{n^3}-\lim_{n\to\infty}\frac{1}{n^3}\sum_{1\le i<j\le n}(-1)^{i+j}\,ij-\lim_{n\to\infty}\frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$2\lim_{n\to\infty}\frac{1}{n^3}\sum_{1\le i<i\le n}(-1)^{i+j}\,ij=-\frac{2}{6}\Rightarrow L=-\frac{1}{6}$$

Solution 3 by Remus Florin Stanca-Romania

Let $n = 2k + 1 \Rightarrow \Omega \overset{\text{Stolz Cesaro}}{=} \lim_{n\to\infty} \frac{\begin{matrix} 1(-2+3-\cdots+n-(n+1))+2(-3+4-\cdots-n+(n+1))+\cdots+ \\ +(n-2)(-(n-1)+n+1+1)+(n-1)(-n+n+1)+n(-(n+1))- \\ -1(-2+\cdots+n)-2(-3+\cdots-n)-\cdots-(n-1)(-(n)) \end{matrix}}{3n^2+3n+1} =$

$$= \lim_{n\to\infty} \frac{(n+1)(-1+2-\cdots-n)}{3n^2+3n+1} =$$

$$= \lim_{n\to\infty} \frac{(n+1)\left(2\left(1+\cdots+\frac{n-1}{2}\right)-(1+3+\cdots+n)\right)}{3n^2+3n+1} =$$

$$= \lim_{n\to\infty} \frac{(n+1)\left(\frac{n-1}{2}\left(\frac{n-1}{2}+1\right)-\left(\frac{n+1}{2}\right)^2\right)}{3n^2+3n+1} =$$

$$= \lim_{n\to\infty} \frac{(n+1)\left(\left(\frac{n-1}{2}\right)^2+\frac{n-1}{2}-\left(\frac{n+1}{2}\right)^2\right)}{3n^2+3n+1} = \lim_{n\to\infty} \frac{(n+1)\left(\frac{n-1}{2}-n\right)}{3n^2+3n+1} =$$

$= \lim_{n\to\infty} -\frac{(n+1)^2}{6n^2+6n+2} = -\frac{1}{6}$ **and if** $n = 2k$ **we have the same result** $\Rightarrow \Omega = -\frac{1}{6}$

Solution 4 by Florentin Vișescu-Romania

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{1\le i<j\le n} (-1)^i i(-1)^j j = \lim_{n\to\infty} \frac{\sum_{1\le i<j\le n}(-1)^i i(-1)^j j}{n^3} \overset{CS}{=}$$

$$\lim_{n\to\infty} \frac{\sum_{1\le i<j\le n}(-1)^i i\,(-1)^j j - \sum_{1\le i<j\le n}(-1)^i i(-1)^j j}{(n+1)^3-n^3} =$$

$$= \lim_{n\to\infty} \frac{\sum_{i=1}^{n}(-1)^i i\,(-1)^{n+1}(n+1)}{3n^2+3n+1} = \lim_{n\to\infty} \underbrace{\frac{(-1)^{n+1}(n+1)\sum_{i=1}^{n}(-1)^i j}{3n^2+3n+1}}_{x_n}$$

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} \frac{(-1)^{2n+1}(2n+1)\sum_{i=1}^{2n}(-1)^i i}{12n^2+6n+1} =$$

$$= \lim_{n\to\infty} \frac{-(2n+1)\left(\overbrace{-1+2}^{1}-\overbrace{3+4}^{1}+\cdots+\overbrace{2n}^{1}\right)}{12n^2+6n+1}$$

$$= \lim_{n\to\infty} \frac{-(2n+1)n}{12n^2+6n+1} = -\frac{1}{6}$$

$$\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} \frac{(-1)^{2n+2}(2n+2)\sum_{i=1}^{2n+1}(-1)^i i}{3(2n+1)^2+3(2n+1)+1}$$

$$= \lim_{n\to\infty} \frac{(2n+2)\big(-1+2-3+4+\cdots+2n-(2n+1)\big)}{3(2n+1)^2+3(2n+1)+1} =$$

$$= \lim_{n\to\infty} \frac{(2n+2)(-n-1)}{3(2n+1)^2+3(2n+1)+1} = -\frac{1}{6}$$

$$\Omega = -\frac{1}{6}$$