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$$\Delta_n = \begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2^3 & 3^3 & \cdots & n^3 \\ 1 & 2^5 & 3^5 & \cdots & n^5 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2^{2n-1} & 3^{2n-1} & \cdots & n^{2n-1} \end{vmatrix}, n \geq 2$$

Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{\Delta_{n+1}}{n^{2n+1}\cdot\Delta_n}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu – Romania, Solution 2 by Remus Florin Stanca – Romania

Solution 1 by Marian Ursărescu – Romania

First, we will prove $\Delta_n = 1!\cdot 3!\cdot\ldots\cdot(2n-1)!\,(1)$ ***by mathematical induction.***

For $P(2): \Delta_2 = \begin{vmatrix} 1 & 2 \\ 1 & 2^3 \end{vmatrix} = 6$ ***true.***

Let $P(n)$ ***true and we will prove*** $P(n+1)$

$$\text{Let } f(x) = \begin{vmatrix} 1 & 2 & \cdots & n & x \\ 1^3 & 2^3 & \cdots & n^3 & x^3 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1^{2n+1} & 2^{2n+1} & \cdots & n^{2n+1} & x^{2n+1} \end{vmatrix}$$

$f\in\mathbb{R}[x]$ ***and grad*** $f = 2n+1$ ***and have*** $0, \pm 1, \pm 2, \ldots, \pm n$ ***roots*** $\Rightarrow$

$$f(x) = a_{2n+1}(x-x_1)(x-x_2)\ldots(x-x_{2n+1}) \Rightarrow$$

$$f(x) = 1!\,3!\cdots(2n-1)!\,x(x^2-1)(x^2-2^2)\cdots(x^2-n^2) \Rightarrow$$

$$\Delta_{n+1} = 1!\,3!\cdots(2n-1)!\,(2n+1)! \Rightarrow P(n+1) \text{ true.}$$

From (1)$\Rightarrow \Omega = \lim_{n\to\infty}\frac{\Delta_{n+1}}{n^{2n+1}\Delta_n} =$

$$= \lim_{n\to\infty}\frac{(2n+1)!}{n^{2n+1}} \quad (2)$$

Let $x_n = \frac{(2n+1)!}{n^{2n+1}}$

$$\lim_{n\to\infty}\frac{x_{n+1}}{x_n} = \lim_{n\to\infty}\frac{(2n+3)!}{(n+1)^{2n+3}}\cdot\frac{n^{2n+1}}{(2n+1)!} =$$

$$= \lim_{n\to\infty} \frac{(2n+3)(2n+2)}{(n+1)^2} \cdot \frac{n^{2n+1}}{(n+1)^{2n+1}} =$$

$$= \lim_{n\to\infty} 4 \cdot \left(\frac{n}{n+1}\right)^{2n+1} = 4 \lim_{n\to\infty} \left[\left(\frac{n}{n+1}\right)^n\right]^{\frac{2n+1}{4}} = \frac{4}{e^2} < 1 \quad (3)$$

From (2)+(3)⇒ $\Omega = 0$.

Solution 2 by Remus Florin Stanca – Romania

$$\Delta_n = n! \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2^2 & 3^2 & \cdots & n^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2^{2n-2} & 3^{2n-2} & \cdots & n^{2n-2} \end{vmatrix}$$

$$= n! \cdot \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 2^2-1 & 3^2-1 & \cdots & n^2-1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2^{2n-2}-1 & 3^{2n-2}-1 & \cdots & n^{2n-2}-1 \end{vmatrix}$$

$$= n! \begin{vmatrix} 2^2-1 & 3^2-1 & \cdots & n^2-1 \\ 2^4-1 & 3^4-1 & \cdots & n^4-1 \\ \cdots & \cdots & \cdots & \cdots \\ 2^{2n-2}-1 & 3^{2n-2} & \cdots & n^{2n-2}-1 \end{vmatrix} =$$

$$= n!\,(2^2-1)(3^2-1)\cdot \ldots \cdot (n^2-1) \cdot$$

$$\cdot \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 2^2+1 & 3^2+1 & \cdots & n^2+1 \\ \cdots & \cdots & \cdots & \cdots \\ 2^{2n-4}+..+1 & 3^{2n-4}+\cdots+1 & \cdots & n^{2n-4}+\cdots+1 \end{vmatrix} =$$

$$= n! \cdot (2^2-1)(3^2-1)\cdot \ldots \cdot (n^2-1) \cdot$$

$$\cdot \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 2^2 & 3^2 & \cdots & n^2 \\ \cdots & \cdots & \cdots & \cdots \\ 2^{2n-4} & 3^{2n-4} & \cdots & n^{2n-4} \end{vmatrix}$$

$$= n! \cdot 2^2 \cdot \ldots \cdot 2^{2n-4} \cdot (2^2-1)(3^2-1)\cdot \ldots \cdot (n^2-1) \cdot$$

$$\cdot \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & \left(\frac{3}{2}\right)^2 & \cdots & \left(\frac{n}{2}\right)^2 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \left(\frac{3}{2}\right)^{2n-4} & \cdots & \left(\frac{n}{2}\right)^{2n-4} \end{vmatrix}$$

$$= n! \cdot 2^2 \cdot \ldots \cdot 2^{2n-4}(2^2-1)(3^2-1)\cdot \ldots \cdot (n^2-1) \cdot$$

$$\cdot\begin{vmatrix}1 & 0 & \cdots & 0\\ 1 & \left(\frac{3}{2}\right)^2-1 & \cdots & \left(\frac{n}{2}\right)^2-1\\ \cdots & \cdots & \cdots & \cdots\\ 1 & \left(\frac{3}{2}\right)^{2n-4}-1 & \cdots & \left(\frac{n}{2}\right)^{2n-4}-1\end{vmatrix}$$

$$= n!\cdot 2^2\cdot\ldots\cdot 2^{2n-4}(2^2-1)(3^2-1)\cdot\ldots\cdot(n^2-1)\cdot$$

$$\cdot\begin{vmatrix}\left(\frac{3}{2}\right)^2-1 & \cdots & \left(\frac{n}{2}\right)^2-1\\ \cdots & \cdots & \cdots\\ \left(\frac{3}{2}\right)^{2n-4}-1 & \cdots & \left(\frac{n}{2}\right)^{2n-4}-1\end{vmatrix}$$

$$= n!\cdot 2^2\cdot\ldots\cdot 2^{2n-4}\cdot(2^2-1)(3^2-1)\cdot\ldots\cdot(n^2-1)\cdot$$

$$\cdot\left(\left(\frac{3}{2}\right)^2-1\right)\cdot\ldots\cdot\left(\left(\frac{n}{2}\right)^2-1\right)\cdot$$

$$\cdot\begin{vmatrix}1 & \cdots & 1\\ \cdots & \cdots & \cdots\\ \left(\frac{3}{2}\right)^{2n-6} & \cdots & \left(\frac{n}{2}\right)^{2n-6}\end{vmatrix} =$$

$$= n!\cdot 2^2\cdot\ldots\cdot 2^{2n-4}\cdot(2^2-1)(3^2-1)\cdot\ldots\cdot(n^2-1)\cdot$$

$$\cdot\left(\left(\frac{3}{2}\right)^2-1\right)\cdot\ldots\cdot\left(\left(\frac{n}{2}\right)^2-1\right)\cdot\left(\frac{3}{2}\right)^2\cdot\ldots\cdot\left(\frac{3}{2}\right)^{2n-6}\cdot$$

$$\cdot\begin{vmatrix}1 & 1 & \cdots & 1\\ 1 & \left(\frac{n}{3}\right)^2 & \cdots & \left(\frac{n}{3}\right)^2\\ \cdots & \cdots & \cdots & \cdots\\ 1 & \left(\frac{4}{3}\right)^{2n-6} & \cdots & \left(\frac{n}{2}\right)^{2n-6}\end{vmatrix}$$

$$= \cdots = n!\left(\left(\left(\frac{2}{1}\right)^2-1\right)\left(\left(\frac{3}{1}\right)^2-1\right)\cdot\ldots\cdot\left(\left(\frac{n}{1}\right)^2-1\right)\cdot\ldots\cdot\left(\left(\frac{n-1}{n-2}\right)^2-1\right)\cdot\left(\left(\frac{n}{n-2}\right)^2-1\right)\right)\cdot$$

$$\cdot\left(\left(\frac{2}{1}\right)^2\cdot\left(\frac{2}{1}\right)^4\cdot\ldots\cdot\left(\frac{2}{1}\right)^{2n-4}\left(\frac{3}{2}\right)^2\cdot\ldots\cdot\left(\frac{3}{2}\right)^{2n-6}\cdot\ldots\cdot\left(\frac{n-2}{n-3}\right)^2\cdot\left(\frac{n-2}{n-3}\right)^4\right)\cdot$$

$$\cdot\begin{vmatrix}1 & 1\\ \left(\frac{n-1}{n-2}\right)^2 & \left(\frac{n}{n-2}\right)^2\end{vmatrix} =$$

$$= n! \cdot \frac{2^{S_{2n-4}} \cdot 3^{S_{2n-6}} \cdot \ldots \cdot (n-2)^{S_4}}{2^{S_{2n-6}} \cdot 3^{S_n-8} \cdot \ldots \cdot (n-3)^{S_4}} \cdot \frac{2n-1}{(n-2)^2} \cdot$$

$$\cdot \frac{(n-1)! \cdot (n-2)! \cdot \ldots \cdot 2!(n+1)!(n+2)! \cdot \ldots \cdot (2n-2)!}{2! \cdot 4! \cdot \ldots \cdot (2n-4)!} \cdot \frac{1}{1^{2n-2} \cdot 2^{2n-4} \cdot \ldots \cdot (n-2)^4} \text{ and } S_{2n} = 2 + 4 + \cdots + 2n$$

$$= n! \cdot 2^{2n-4} \cdot 3^{2n-6} \cdot \ldots \cdot (n-3)^6 (n-2)^6 \cdot \frac{(2n-1)}{(n-2)^2} \cdot$$

$$\cdot \frac{(n-1)!\,(n-2)! \cdot \ldots \cdot 2!\,(n+1)!\,(n+2)! \cdot \ldots \cdot (2n-2)!}{2! \cdot 4! \cdot \ldots \cdot (2n-4)!} \cdot \frac{1}{2^{2n-4} \cdot \ldots \cdot (n-2)^4} =$$

$$= n! \cdot (n-2)^2 \cdot \frac{(2n-1)}{(n-2)^2} \cdot (n-1)! \cdot (n-2)! \cdot \ldots \cdot 2! \cdot$$

$$\cdot (n+1)!\,(n+2)! \cdot \ldots \cdot (2n-2)! \cdot \frac{1}{2! \cdot 4! \cdot \ldots \cdot (2n-4)!}$$

$$= \frac{2! \cdot 3! \cdot \ldots \cdot (2n-2)! \cdot (2n-1)}{2! \cdot 4! \cdot \ldots \cdot (2n-4)} =$$

$$= 1! \cdot 3! \cdot 5! \cdot \ldots \cdot (2n-1)! = \Delta_n$$

$$\Rightarrow \Omega = \lim_{n\to\infty} \frac{1! \cdot 3! \cdot \ldots \cdot (2n+1)!}{n^{2n+1} \cdot 1 \cdot 3! \cdot \ldots \cdot (2n-1)!}$$

$$= \lim_{n\to\infty} \frac{(2n+1)!}{n^{2n+1}} = ?$$

$$\text{Let } x_n = \frac{(2n+1)!}{n^{2n+1}} \Rightarrow \frac{x_{n+1}}{x_n} = \frac{(2n+3)!}{(n+1)^{2n+3}} \cdot \frac{n^{2n+1}}{(2n+1)!} \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} (2n+2)(2n+3) \cdot \frac{1}{e^2} \cdot \frac{1}{(n+1)^2} =$$

$$= \lim_{n\to\infty} \frac{n}{e^2} < 1 \Rightarrow \lim_{n\to\infty} x_n = 0 \Rightarrow \Omega = 0$$