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If $\frac{\sqrt{3}}{3} < a \leq b < 1$ **then:**

$$\int_a^b \int_a^b \left(\frac{x+y}{\tan^{-1}\left(\frac{x+y}{2}\right)}\right) dx\, dy \geq \frac{(b^2-a^2)(b-a)}{2\tan^{-1}\left(\frac{a+b}{2}\right)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ali Jaffal-Lebanon, Solution 2 by Tran Hong-Dong Thap-Vietnam

Solution 1 by Ali Jaffal-Lebanon

Let $\varphi(t) = \frac{t}{\arctan(t)}$ ***where*** $t > 0$

$$\varphi''(t) = \frac{2(t-\arctan t)}{(\arctan t)^3(1+t^2)^2} > 0$$

Since $t \geq \arctan t$ ***for all*** $t \in [0,+\infty[$ ***we have*** φ ***is convexe on*** $]0,+\infty[$

then $\int_a^b \varphi(t)dt \geq (v-u)\varphi\left(\frac{v+u}{2}\right)$ ***for all*** $0 < u < v$

Let $I = \int_a^b \int_a^b \frac{x+y}{\arctan\left(\frac{x+y}{2}\right)} dx\, dy = 2\int_a^b \int_a^b \varphi\left(\frac{x+y}{2}\right) dx\, dy$

$$\int_a^b \varphi\left(\frac{x+y}{2}\right) dx \geq (b-a)\varphi\left(\frac{b+a}{4}+\frac{y}{2}\right)$$

then

$$I \geq 2\int_a^b (b-a)\varphi\left(\frac{b+a}{4}+\frac{y}{2}\right)dy \geq 2(b-a)(b-a)\varphi\left(\frac{b+a}{4}+\frac{b+a}{4}\right)$$

$$\geq 2(b-a)(b-a)\cdot\varphi\left(\frac{b+a}{2}\right) \geq 2(b-a)(b-a)\times\frac{\frac{b+a}{2}}{\tan^{-1}\left(\frac{b+a}{2}\right)}$$

$$\geq \frac{(b-a)(b^2-a^2)}{\tan^{-1}\left(\frac{a+b}{2}\right)} \geq \frac{(b-a)(b^2-a^2)}{2\tan^{-1}\left(\frac{a+b}{2}\right)}$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

With $\frac{\sqrt{3}}{3} < a \leq b < 1$ **and** $x, y \in [a; b] \to a \leq \frac{x+y}{2} \leq b$. **We have:**

$$f(t) = \frac{t}{\tan^{-1}t}, t > 0 \to f'(t) = \frac{1}{\tan^{-1}t} - \frac{t}{(t^2+1)(\tan^{-1}t)^2} = \frac{\tan^{-1}t - \frac{t}{(t^2+1)}}{(\tan^{-1}t)^2}$$

Let: $g(t) = \tan^{-1}t - \frac{t}{(t^2+1)}, t > 0 \to g'(t) = \frac{2t^2}{(t^2+1)^2} > 0 \to g(t) \uparrow$ **on** $(0; +\infty) \to$

$$g(t) > g(0) = 0 \to f'(t) > 0 \to f(t) \uparrow \text{ on } (0; +\infty)$$

Choosing: $t = \frac{x+y}{x} \geq a > 0 \to \frac{\frac{x+y}{x}}{\tan^{-1}\left(\frac{x+y}{2}\right)} \geq \frac{a}{\tan^{-1}(a)}$

We must show that: $\frac{a}{\tan^{-1}(a)} \geq \frac{b+a}{4\tan^{-1}\left(\frac{a+b}{2}\right)} \leftrightarrow 4a\tan^{-1}\left(\frac{a+b}{2}\right) \geq (b+a)\tan^{-1}(a)$

It is true because: $\tan^{-1}x \uparrow$ **on** $(0, +\infty) \overset{\frac{a+b}{2}>0}{\to} \tan^{-1}\left(\frac{a+b}{2}\right) \geq \tan^{-1}(a)$

And: $4a \geq b + a \leftrightarrow 3a \overset{(*)}{\geq} b$

$a > \frac{\sqrt{3}}{3} \to 3a > \sqrt{3} > 1 > b$. **So,**

$$\int_a^b\int_a^b\left(\frac{\frac{x+y}{2}}{\tan^{-1}\left(\frac{x+y}{2}\right)}\right)dx\,dy \geq \left(\frac{b+a}{4\tan^{-1}\left(\frac{a+b}{2}\right)}\right)\int_a^b\int_a^b dx\,dy = \frac{(b-a)^2(b+a)}{4\tan^{-1}\left(\frac{a+b}{2}\right)}$$

Or

$$\int_a^b\int_a^b\left(\frac{x+y}{\tan^{-1}\left(\frac{x+y}{2}\right)}\right)dx\,dy \geq \frac{(b^2-a^2)(b-a)}{2\tan^{-1}\left(\frac{a+b}{2}\right)}$$