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If

$$f(y)=\int_0^1 \frac{\sin^{-1}(xy)}{\sqrt{1-x^2}}dx$$

then show that

$$\int_0^1 f(y)\,dy=\frac{\pi^2}{8}-\log(2)\,;\int_0^1 \frac{f(y)}{y}dy=\frac{7\zeta(3)}{8}$$

$$\int_0^1 \frac{f(y)}{\sqrt{y}}dy=\frac{\pi^2}{4}-\pi+2\log(2)$$

$$\int_0^1 f(y)\log(y)\,dy=2\log(2)-\frac{\pi^2}{6}$$

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Solution 1 by Mokhtar Khassani-Mostaganem-Algerie, Solution 2 by Tobi Joshua-Nigeria

Solution 1 by Mokhtar Khassani-Mostaganem-Algerie

$$f(y)=\int_0^1 \frac{\arcsin(xy)}{\sqrt{1-x^2}}dx=\int_0^{\frac{\pi}{2}}\arcsin(y\sin x)\,dx=\sum_{n=0}^{\infty}\frac{y^{2n+1}\binom{2n}{n}}{(2n+1)4^n}\int_0^{\frac{\pi}{2}}\sin^{2n+1}x\,dx=$$

$$=\sum_{n=0}^{\infty}\frac{y^{2n+1}\binom{2n}{n}}{(2n+1)4^n}\cdot\frac{\sqrt{\pi}\Gamma(n+1)}{2\Gamma\left(n+1+\frac{1}{2}\right)}=\sum_{n=0}^{\infty}\frac{y^{2n+1}\binom{2n}{n}}{(2n+1)4^n}\cdot\frac{\sqrt{\pi}n!}{2\,\frac{(2n+2)!\sqrt{\pi}}{4^{n+1}(n+1)!}}=$$

$$=\sum_{n=0}^{\infty}\frac{y^{2n+1}}{(2n+1)^2}=\frac{Li_2(y)-Li_2(-y)}{2}=$$

$$M=\int_0^1 f(y)\,dy=\int_0^1 \frac{Li_2(y)-Li_2(-y)}{2}dy=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\int_0^1 y^{2n+1}\,dy=$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2(n+1)}=\sum_{n=0}^{\infty}\left(\frac{1}{(2n+1)^2}+\frac{1}{2(n+1)}-\frac{1}{2n+1}\right)=\frac{\pi^8}{8}-\log 2$$

$$N=\int_0^1\frac{f(y)}{y}dy=\int_0^1\frac{Li_2(y)-Li_2(-y)}{2y}dy=\frac{1}{2}\{Li_3(x)-Li_3(-x)\}_0^1=$$

$$=\frac{1}{2}\left(\zeta(3)+\frac{3}{4}\zeta(3)\right)=\frac{7}{8}\zeta(3)$$

$$P=\int_0^1\frac{f(y)}{\sqrt{y}}dy=\int_0^1\frac{Li_2(y)-Li_2(-y)}{2\sqrt{y}}dy=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\int_0^1 y^{2n+\frac{1}{2}}dy=$$

$$=2\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2(4n+3)}=\sum_{n=0}^{\infty}\left(\frac{2}{(2n+1)^2}-\frac{1}{2\left(n+\frac{1}{2}\right)\left(n+\frac{3}{4}\right)}\right)=$$

$$=\frac{\pi^2}{4}-\frac{1}{2}\cdot\frac{\Psi\left(\frac{1}{2}\right)-\Psi\left(\frac{3}{4}\right)}{\frac{1}{2}-\frac{3}{4}}=\frac{\pi^2}{4}-\pi+2\log 2$$

$$\Omega=\int_0^1 f(y)\log y\,dy=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\int_0^1 y^{2n+1}\log y\,dy=$$

$$=-\sum_{n=0}^{\infty}\frac{1}{4(2n+1)^2(n+1)^2}=\sum_{n=0}^{\infty}\left(\frac{1}{2\left(n+\frac{1}{2}\right)(n+1)}-\frac{1}{(2n+1)^2}-\frac{1}{4(n+1)^2}\right)=$$

$$=\frac{1}{2}\cdot\frac{\Psi\left(\frac{1}{2}\right)-\Psi(1)}{\frac{1}{2}-1}-\frac{\pi^2}{8}-\frac{\pi^2}{24}=2\log 2-\frac{\pi^2}{6}$$

Solution 2 by Tobi Joshua-Nigeria

$$f(y)=\int_0^1\frac{\sin^{-1}(xy)}{\sqrt{1-x^2}}dx,\ x=\sin\theta,\ f(y)=\int_0^{\frac{\pi}{2}}\sin^{-1}(y\sin\theta)\,d\theta$$

$$f(y)=\int_0^{\frac{\pi}{2}}\int_0^{y}\frac{d\theta\,dz}{\sqrt{\left(\frac{1}{\sin\theta}\right)^2-z^2}},\ f(y)=\int_0^{y}\frac{dz}{z}\int_0^{\frac{\pi}{2}}\frac{\sin\theta\,d\theta}{\sqrt{\frac{1-z^2}{z^2}+(\cos\theta)^2}}$$

$$f(y)=-\int_0^{y}\frac{dz}{z}\int_0^{\frac{\pi}{2}}\frac{d(\cos\theta)}{\sqrt{\left(\frac{\sqrt{1-z^2}}{z}\right)^2+(\cos\theta)^2}},\ f(y)=\int_0^{y}\frac{dz}{z}\sinh^{-1}\left(\frac{z}{\sqrt{1-z^2}}\right)$$

$$f(y)=\int_0^y\frac{dz}{z}\log\left(\frac{z}{\sqrt{1-z^2}}+\sqrt{\frac{z^2}{1-z^2}+1}\right),\ f(y)=\int_0^y\frac{dz}{z}\log\left(\frac{z+1}{\sqrt{1-z^2}}\right)$$

$$f(y)=\frac{1}{2}\int_0^y\frac{dz}{z}\log\left(\frac{z+1}{1-z}\right),\ f(y)=\frac{1}{2}\int_0^y\frac{\log(1+z)-\log(1-z)}{z}dz$$

$$f(y)=\frac{1}{2}[Li_2(y)-Li_2(-y)],\ f(y)=\frac{1}{2}[Li_2(y)-Li_2(-y)]\quad (*)$$

$$(1)\ A=\int_0^1 f(y)\,dy$$

$$A=\frac{1}{2}[Li_2(y)-Li_2(-y)]dy$$

$$A=\frac{1}{2}\left[y\big(Li_2(y)-Li_2(-y)\big)_0^1+\int_0^1\ln(1-y)-\int_0^1\ln(1+y)\,dy\right]$$

$$A=\frac{1}{2}\left[\frac{\pi^2}{4}+\left(-\ln(1-y^2)+y\ln\left(\frac{1-y}{1+y}\right)\right)_0^1\right],\ A=\frac{1}{2}\left[\frac{\pi^2}{4}-2\ln(2)\right]$$

$$A=\left[\frac{\pi^2}{8}-\ln(2)\right],\ (2)$$

$$B=\int_0^1\frac{f(y)}{y}dy,\ B=\frac{1}{2}\int_0^1\frac{\big(Li_2(y)-Li_2(-y)\big)}{y}dy$$

$$B=\frac{1}{2}\left[\log(y)\big(Li_2(y)-Li_2(-y)\big)_0^1+\int_0^1\frac{\ln(y)}{y}\ln(1-y)-\int_0^1\frac{\ln(y)}{y}\ln(1+y)\,dy\right]$$

$$B=\frac{1}{2}\cdot\frac{\partial}{\partial a}\bigg]_{a=0}\left[\int_0^1 y^{a-1}\ln(1-y)\,dy-\int_0^1 y^{a-1}\ln(1+y)\,dy\right]$$

$$B=\frac{1}{2}\cdot\frac{\partial}{\partial a}\bigg]_{a=0}\sum_{k=0}^{\infty}\frac{1}{(k+1)}\left[-\int_0^1 y^a y^k dy-(-1)^k\int_0^1 y^a y^k\,dy\right]$$

$$B=\frac{1}{2}\cdot\frac{\partial}{\partial a}\bigg]_{a=0}\sum_{k=0}^{\infty}\frac{1}{(k+1)}\left[-\frac{1}{(a+k+1)}-\frac{(-1)^k}{(a+k+1)}\right]$$

$$B=\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(k+1)}\left[\frac{1}{(k+1)^2}+\frac{(-1)^k}{(k+1)^2}\right],\ B=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{(k)}\left[\frac{1}{(k)^2}+\frac{(-1)^{k-1}}{(k)^2}\right]$$

$$B=\frac{1}{2}[\zeta(3)+\eta(3)],\ B=\frac{1}{2}\left[\zeta(3)+\frac{3}{4}\zeta(3)\right]$$

$$B=\left[\frac{7}{8}\zeta(3)\right]\otimes$$

(3) $C=\int_0^1 \frac{f(y)}{\sqrt{y}}dy$

$$C=\frac{1}{2}\int_0^1 \frac{\left(Li_2(y)-Li_2(-y)\right)}{\sqrt{y}}dy$$

$$C=\frac{1}{2}\left[2\sqrt{y}\left(\left(Li_2(y)-Li_2(-y)\right)_0^1+2\int_0^1 \frac{1}{\sqrt{y}}\ln(1-y)-2\int_0^1 \frac{1}{\sqrt{y}}\ln(1+y)\,dy\right)\right]$$

$$C=\frac{1}{2}\left[\frac{\pi^2}{2}+2\int_0^1 \frac{1}{\sqrt{y}}\ln(1-y)-2\int_0^1 \frac{1}{\sqrt{y}}\ln(1+y)\,dy\right]$$

$$C=\frac{\pi^2}{4}+\frac{1}{2}\left[2\int_0^1 y^{-\frac{1}{2}}\ln\left(\frac{1-y}{1+y}\right)dy\right],\ C=\frac{\pi^2}{4}+\frac{1}{2}\left[4\int_0^1 \ln\left(\frac{1-y^2}{1+y^2}\right)dy\right]$$

$$C=\frac{\pi^2}{4}+2\left[y\ln\left(\frac{1-y^2}{1+y^2}\right)+2\int_0^1 \frac{y^2dy}{1-y^2}+2\int_0^1 \frac{y^2dy}{1+y^2}\right]$$

$$C=\frac{\pi^2}{4}+2\left[y\ln\left(\frac{1-y^2}{1+y^2}\right)+2\left(-y+\frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)\right)+2(y-\tan^{-1}y)\right]_0^1$$

$$C=\frac{\pi^2}{4}+2[(y-1)\ln(1-y)+(y+1)\ln(1+y)-y\ln(1+y^2)-2\tan^{-1}y]_0^1$$

$$C=\frac{\pi^2}{4}+2\left[(2)\ln(2)-\ln(2)-\frac{\pi}{2}\right],\ C=\frac{\pi^2}{4}+(2)\ln(2)-\pi \quad \otimes$$

(4) $D=\int_0^1 f(y)\ln y\,dy$

$$D=\frac{1}{2}\int_0^1 \ln y\left(Li_2(y)-Li_2(-y)\right)dy,\ D=\frac{1}{2}\cdot\frac{\partial}{\partial s}\Big]_{s=0}\left[\int_0^1 y^2\left(Li_2(y)-Li_2(-y)\right)dy\right]$$

$$D=\frac{1}{2}\cdot\frac{\partial}{\partial s}\Big]_{s=0}\left[\frac{y^{s+1}}{s+1}\left(Li_2(y)-Li_2(-y)\right)_0^1+\int_0^1 \frac{y^s}{s+1}\ln\left(\frac{1-y}{1+y}\right)dy\right]$$

$$D=\frac{1}{2}\cdot\frac{\partial}{\partial s}\Big]_{s=0}\left[\frac{1}{s+1}\cdot\frac{\pi^2}{4}+\int_0^1 \frac{y^s}{s+1}\ln(1-y)\,dy-\frac{1}{s+1}\int_0^1 y^s\ln(1+y)\,dy\right]$$

$$D=\frac{1}{2}\cdot\frac{\partial}{\partial s}\Big]_{s=0}\left[\frac{\pi^2}{4(s+1)}+\sum_{k=0}^{\infty}\frac{1}{(k+1)}\left[-\frac{1}{s+1}\int_0^1 y^{s+k+1}\,dy-(-1)^k\int_0^1 y^{s+k+1}\,dy\right]\right.$$

$$D=\frac{1}{2}\cdot\frac{\partial}{\partial s}\Big]_{s=0}\left[\frac{\pi^2}{4(s+1)}+\sum_{k=0}^{\infty}\frac{1}{(k+1)}\left[-\frac{1}{(s+1)(k+s+2)}-\frac{(-1)^k}{(s+1)(s+k+2)}\right]\right]$$

$$D=\left[-\frac{\pi^2}{8}+\sum_{k=0}^{\infty}\frac{1}{(k+1)}\left[\frac{1}{(k+2)}+\frac{1}{(k+2)^2}+\frac{(-1)^k}{(k+2)}+\frac{(-1)^k}{(k+2)^2}\right]\right]$$

$$D=\left[-\frac{\pi}{8}+\frac{1}{2}\left[(\psi(2)+\gamma)+\left(2-\frac{\pi^2}{6}\right)+(\ln 2)-(1-\ln 2)+\left(-2+\frac{\pi^2}{12}+2\ln 2\right)\right]\right]$$

$$D=\left[-\frac{\pi^2}{8}+\frac{1}{2}+\left(1-\frac{\pi^2}{12}\right)-\frac{1}{2}+\ln 2+\left(-1+\frac{\pi^2}{25}+\ln 2\right)\right]$$

$$D=\left[-\frac{\pi^2}{6}+2\ln 2\right]\ (*)$$