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In ΔABC, g_a - Gergonne's cevian the following relationship holds:

$$3 + \frac{1}{r}\sum_{cyc}(w_a - g_a) \geq \frac{w_a}{h_a} + \frac{w_b}{h_b} + \frac{w_c}{h_c}$$

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$$\text{Firstly, } w_a = r\frac{w_a}{h_a} + AI \Leftrightarrow w_a = \frac{ar}{2rs}w_a + AI \Leftrightarrow w_a\left(1 - \frac{a}{2s}\right) = AI$$

$$\Leftrightarrow \frac{2bc}{b+c}\cos\frac{A}{2}\left(\frac{b+c}{2s}\right) = \frac{r}{\sin\frac{A}{2}} \Leftrightarrow \frac{bc}{2s}\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right) = r \Leftrightarrow \frac{bc}{2s}\left(\frac{a}{2R}\right) = r \Leftrightarrow \frac{4Rrs}{4RS} = r \to \text{true}$$

$$\therefore w_a \overset{(1)}{=} r\left(\frac{w_a}{h_a}\right) + AI. \text{ Similarly, } w_b \overset{(2)}{=} r\left(\frac{w_b}{h_b}\right) + BI \text{ and } w_c \overset{(3)}{=} r\left(\frac{w_c}{h_c}\right) + CI$$

$$(1)+(2)+(3) \Rightarrow \sum w_a \overset{(a)}{=} r\left(\sum\frac{w_a}{h_a}\right) + \sum AI$$

$$\text{Now, triangle inequality} \Rightarrow g_a \leq AI + r \text{ and analogs} \Rightarrow \sum g_a \leq \sum AI + 3r$$

$$\Rightarrow \frac{1}{r}\sum g_a \overset{(b)}{\leq} \frac{1}{r}\sum AI + 3$$

$$(a), (b) \Rightarrow \sum\frac{w_a}{h_a} + \frac{1}{r}\sum g_a \leq \frac{1}{r}\sum w_a - \frac{1}{r}\sum AI + \frac{1}{r}\sum AI + 3 \Rightarrow 3 + \frac{1}{r}\sum(w_a - g_a) \geq \sum\frac{w_a}{h_a}$$

(Proved)