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In $\Delta ABC, I$ – incenter, the following relationship holds:

$$r\left(\sum_{cyc}(m_a + r_a)\right)\left(\sum_{cyc}(IA + r_a)\right) \le sR\sqrt{3}\left(\sum_{cyc}(m_a + h_a) - 3r\right)$$

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$$\text{Firstly, } s^2 - 2Rr + r^2 \ge \sqrt{8Rr}\sqrt{s^2 + 4Rr + r^2}$$

$$\Leftrightarrow (s^2 - 2Rr + r^2)^2 \ge 8Rr(s^2 + 4Rr + r^2)$$

$$\Leftrightarrow s^4 + (2Rr - r^2)^2 - 2s^2(2Rr - r^2) \ge 8Rrs^2 + 8Rr(4Rr + r^2)$$

$$\Leftrightarrow s^4 \overset{(1)}{\ge} s^2(12Rr - 2r^2) + 8Rr(4Rr + r^2) - (2Rr - r^2)^2$$

$$\text{Now, LHS of (1)} \overset{Gerretsen}{\ge} s^2(16Rr - 5r^2)$$

$$\overset{?}{\ge} s^2(12Rr - 2r^2) + 8Rr(4Rr + r^2) - (2Rr - r^2)^2$$

$$\Leftrightarrow s^2(4Rr - 3r^2) \underset{(2)}{\overset{?}{\ge}} 8Rr(4R + r^2) - (2Rr - r^2)^2$$

$$\text{Again, LHS of (2)} \overset{Gerretsen}{\ge} (16Rr - 5r^2)(4Rr - 3r^2)$$

$$\overset{?}{\ge} 8Rr(4Rr + r^2) - (2Rr - r^2)^2 \Leftrightarrow 9R^2 - 20Rr + 4r^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow (R - 2r)(9R - 2r) \overset{?}{\ge} 0 \to \text{true} \because R \overset{Euler}{\ge} 2r$$

$$\therefore s^2 - 2Rr + r^2 \overset{(a)}{\geq} \sqrt{8Rr}\sqrt{s^2 + 4Rr + r^2}$$

Secondly, $\frac{b+c}{2} \geq \sqrt{2r(r_b + r_c)} \Leftrightarrow \frac{(b+c)^2}{4} \geq 2r\left(\frac{rs}{s-b} + \frac{rs}{s-c}\right)$

$$\Leftrightarrow \frac{(b+c)^2}{4} \geq 2r^2 s\left(\frac{2s-b-c}{(s-b)(s-c)}\right) = \frac{2a(s-a)(s-b)(s-c)}{(s-b)(s-c)}$$

$$\Leftrightarrow (b+c)^2 \geq 4a(b+c-a) \Leftrightarrow (b+c)^2 + 4a^2 - 4a(b+c) \geq 0$$

$$\Leftrightarrow (b+c-2a)^2 \geq 0 \rightarrow \textit{true} \therefore \frac{b+c}{2} \geq \sqrt{2r(r_b + r_c)}$$

$$= \sqrt{2rs\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)} = \sqrt{2rs\left(\frac{\sin\frac{B+C}{2}\cos\frac{A}{2}}{\prod\cos\frac{A}{2}}\right)}$$

$$= \sqrt{\left(\frac{2rs}{\left(\frac{s}{4R}\right)}\right)\cos^2\frac{A}{2}} = \sqrt{8Rr}\cos\frac{A}{2} \therefore \frac{b+c}{2} \overset{(b)}{\geq} \sqrt{8Rr}\cos\frac{A}{2}$$

Thirdly, $\sum AI = r\sum\sqrt{\frac{bc}{(s-b)(s-c)}} = r\sum\sqrt{\frac{bc(s-a)}{r^2 s}} \overset{CBS}{\leq} \frac{\sqrt{\sum ab}\sqrt{\sum(s-a)}}{\sqrt{s}} = \sqrt{\sum ab}$

$$\therefore \sum AI \overset{(c)}{\leq} \sqrt{s^2 + 4Rr + r^2}$$

Now, $\sum h_a - 3r = \frac{s^2+4Rr+r^2}{2R} - 3r = \frac{s^2-2Rr+r^2}{2R} \overset{by\ (a)}{\geq} \frac{\sqrt{8Rr}\sqrt{s^2+4Rr+r^2}}{2R}$

$$\overset{by\ (c)}{\geq} \frac{\sqrt{8Rr}\sum AI}{2R} = \sqrt{\frac{2r}{R}}\sum AI \Rightarrow sR\sqrt{3}\left(\sum h_a - 3r\right) \geq sR\sqrt{3}\left(\sqrt{\frac{2r}{R}}\sum AI\right)$$

$$\Rightarrow sR\sqrt{3}\left(\sum h_a - 3r\right) \overset{(d)}{\geq} (s\sqrt{3})(\sqrt{2Rr})\left(\sum AI\right)$$

Also, $sR\sqrt{3}(\sum m_a) \overset{Ioscu}{\geq} sR\sqrt{3}\left(\sum\frac{b+c}{2}\cos\frac{A}{2}\right) \overset{by\ (b)}{\geq} sR\sqrt{3}\left(\sum\sqrt{8Rr}\cos^2\frac{A}{2}\right)$

$$= (sR\sqrt{3})(\sqrt{2Rr})\left(\sum(1+\cos A)\right) = (sR\sqrt{3})(\sqrt{2Rr})\left(3+1+\frac{r}{R}\right)$$

$$= (s\sqrt{3})(\sqrt{2Rr})(4R+r) \therefore sR\sqrt{3}\left(\sum m_a\right) \overset{(e)}{\geq} (s\sqrt{3})(\sqrt{2Rr})\left(\sum r_a\right)$$

(d)+(e)$\Rightarrow sR\sqrt{3}(\sum(m_a + h_a) - 3r) \overset{(i)}{\geq} (s\sqrt{3})(\sqrt{2R})(\sum(IA + r_a))$

Moreover, $r(\sum(m_a + r_a))(\sum(IA + r_a)) \leq r(\sum(r_a + r_a))(\sum(IA + r_a))$

$$= 2r(4R + r)\left(\sum(IA + r_a)\right)$$

$$\Rightarrow r\left(\sum(m_a + r_a)\right)\left(\sum(IA + r_a)\right) \overset{(ii)}{\leq} 2r(4R + r)\left(\sum(IA + r_a)\right)$$

(i), (ii)* ⇒ *it suffices to prove:

$$(s\sqrt{3})(\sqrt{2Rr})\left(\sum(IA + r_a)\right) \geq 2r(4R + r)\left(\sum(IA + r_a)\right)$$

$$\Leftrightarrow 2Rr(3s^2) \geq 4r^2(4R + r)^2 \Leftrightarrow 3Rs^2 \overset{(m)}{\geq} 2r(4R + r)^2$$

Now, LHS of (m) $\overset{Gerretsen}{\geq} 3R(16Rr - 5r^2) \overset{?}{\geq} 2r(4R + r)^2$

$$\Leftrightarrow 16R^2 - 31Rr - 2r^2 \overset{?}{\geq} 0 \Leftrightarrow (R - 2r)(16R + r) \overset{?}{\geq} 0 \rightarrow$$ ***true*** $\because R \overset{Euler}{\geq} 2r$

⇒ ***(m)*** ⇒ ***proposed inequality is true (Proved)***