R M M

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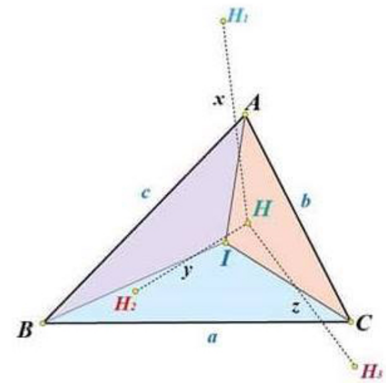
H – orthocenter of ABC, I incenter of ABC. H_1, H_2, H_3 are orthocenters of BIC, CIA and AIB. a, b, c side lengths, R circumradius of ABC.

$HH_1 = x, HH_2 = y, HH_3 = z$. Prove that:

$$xy + yz + zx + ab + bc + ca \leq 12R^2$$



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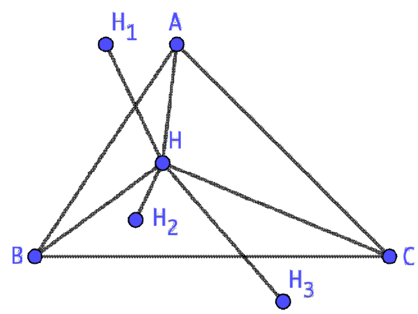

***Let** $z_1, z_2, z_3 \in \mathbb{C}$ **such that** $A(z_1), B(z_2), C(z_3), \Delta ABC \subset C(O, R)$*

$$\left.\begin{matrix} z_H = z_A + z_B + z_C \\ z_{H_1} = z_b + z_C + z_I \end{matrix}\right\} \Rightarrow x = HH_1 = |z_H - z_{H_1}| = |z_A - z_I| = AI \Rightarrow \text{we must show:}$$

$$AI \cdot BI + BI \cdot CI + CI \cdot AI + ab + ac + bc \leq 12R^2 \quad (1)$$

But $AI \cdot BI + BI \cdot CI + CI \cdot AI \leq AI^2 + BI^2 + CI^2$ **and** $AI^2 + BI^2 + CI^2 =$

$$= s^2 + r^2 - 8Rr \Rightarrow AI \cdot BI + BI \cdot CI + CI \cdot AI \leq s^2 + r^2 - 8Rr \quad (2)$$

$$ab + ac + bc = s^2 + r^2 + 4Rr \quad (3)$$

$$2s^2 + 2r^2 - 4Rr \leq 12R^2 \quad (4)$$

From (1)+(2)+(3) we must show: from Carlitz inequality: $2s^2 \leq 9R^2 + 8Rr + 2r^2 \Rightarrow$

$\Rightarrow 2s^2 + 2r^2 - 4Rr \leq 9R^2 + 4Rr + 4r^2$ ***(5)***

From (4)+(5) we must show: $9R^2 + 4Rr + 4r^2 \leq 12R^2 \Leftrightarrow 4Rr + 4r^2 \leq 3R^2$ ***(true, because*** $4r^2 \leq R^2$ ***and*** $4Rr \leq 2R^2$***)***